Exercises Chapter 1

Mathematical Methods of Bioengineering

February 9, 2021

Contents

This is a guide to the exercises that you can solve. If you fall short, there are more similar exercises in the books of the subject. You can ask me questions at the end of class or in tutoring.

The idea is that you solve the exercises and do them on the blackboard in class. Each time you go to the board, it will count to the 5% of the final mark. You must participate at least 3 times in order to get the full 5% and at least 6 times to raise the final grade by +0.5 points.

1 Vectors

1.1 Section 1.1

1. Let A be the point with coordinates $(1, 0, 2)$, let B be the point with coordinates $(-3, 3, 1)$, and let C be the point with coordinates $(2, 1, 5)$.

   (a) Describe the vectors $\vec{AC}$, $\vec{CB}$ and $\vec{AB}$.

   (b) Explain, with pictures, why $\vec{AC} + \vec{CB} = \vec{AB}$.

2. Show that the length of the vector (3,1) is $\sqrt{10}$ using the Pythagorean Theorem.

3. Let $\mathbf{a}$=(a_1, a_2, a_3) and $\mathbf{b}$=(b_1, b_2, b_3) be two nonzero vectors such that $\mathbf{b} \neq k\mathbf{a}$. Use vectors to describe the set of points inside the parallelogram with vertex $P_0 = (x_0, y_0, z_0)$ and whose adjacent sides are parallel to $\mathbf{a}$ and $\mathbf{b}$ and have the same lengths as $\mathbf{a}$ and $\mathbf{b}$. Figure in the next page.

4. A flea falls onto marked graph paper at the point (3, 2). She begins moving from that point with velocity vector $v = (-1, -2)$ (i.e., she moves 1 graph paper unit per minute in the negative x-direction and 2 graph paper units per minute in the negative y-direction).

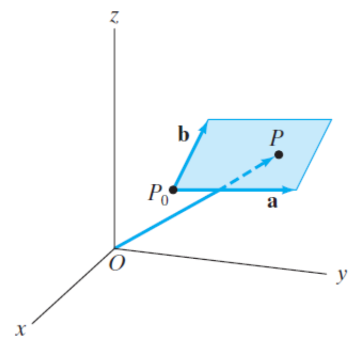


Figure 1: Figure exercises 3.

(a) What is the speed of the flea?

(b) Where is the flea after 3 minutes?

(c) How long does it take the flea to get to the point (-4,-12)?

(d) Does the flea reach the point (-13,-27)? Why or why not?

5. A plane takes off from an airport with velocity vector (50, 100, 4). Assume that the units are miles per hour, that the positive x-axis points east, and that the positive y-axis points north.

   (a) How fast is the plane climbing vertically at takeoff?

   (b) Suppose the airport is located at the origin and a skyscraper is located 5 miles east and 10 miles north of the airport. The skyscraper is 1,250 feet tall. When will the plane be directly over the building?

   (c) When the plane is over the building, how much vertical clearance is there?

   Note: 1 Mile = 5280 feet.

1.2 Section 1.2

1. Let $\mathbf{a_1} = (1, 1)$ and $\mathbf{a_2} = (1, -1)$.

   (a) Write the vector $\mathbf{b} = (3, 1)$ as $c_1\mathbf{a_1} + c_2\mathbf{a_2}$, where c_1 and c_2 are appropriate scalars.

   (b) Repeat part (a) for the vector $\mathbf{b} = (3, -5)$.

   (c) Show that any vector $\mathbf{b} = (b_1, b_2)$ in $\mathbb{R}^2$ may be written in the form $c_1\mathbf{a_1} + c_2\mathbf{a_2}$ for appropriate choices of the scalars c_1, c_2. (This shows that a_1 and a_2 form a basis for $\mathbb{R}^2$ that can be used instead of $\mathbf{i}$ and $\mathbf{j}$).

2. Let $\mathbf{a_1} = (1, 0, -1)$, $\mathbf{a_2} = (0, 1, 0)$ and $\mathbf{a_3} = (1, 1, -1)$

   (a) Find scalars c_1, c_2, c_3 so as to write the vector $\mathbf{b} = (5, 6, -5)$ as $c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3}$.

   (b) Try to do the same with $\mathbf{b} = (2, 3, 4)$. What happens? Can the vectors be a basis for $\mathbb{R}^3$?

3. Write a set of parametric equations for the line in $\mathbb{R}^5$ through the points $(9, \pi, -1, 5, 2)$ and $(-1, 1, 2, 7, 1)$.

4. (a) Write a set of parametric equations for the line in $\mathbb{R}^3$ through the point $(-1, 7, 3)$ and parallel to the vector $(2, -1, 5)$.
   (b) Write different (but equally correct) sets of equations for parts (a).
   (c) Find the symmetric forms of your answers.

5. Show that the two sets of equations

$$\frac{x-2}{3} = \frac{y-1}{7} = \frac{z}{5} \quad \text{and} \quad \frac{x+1}{-6} = \frac{y+6}{-14} = \frac{z+5}{-10}$$

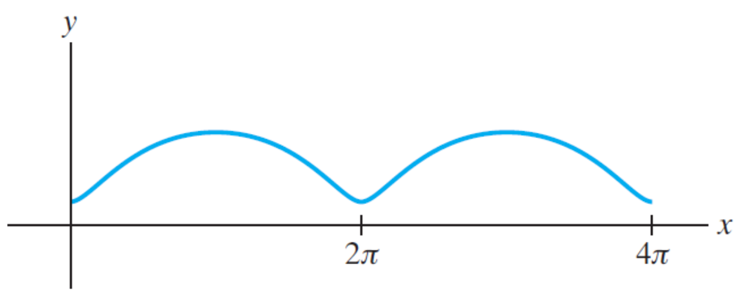
   actually represents the same line in $\mathbb{R}^3$.

6. Show that the line with symmetric form

$$\frac{x-3}{-2} = y - 5 = \frac{z+2}{3}$$

   lies entirely in the plane $3x + 3y + z = 22$.

7. Suppose that a bicycle wheel of radius a rolls along a flat surface without slipping. If a reflector is attached to a spoke of the wheel at a distance b from the center, the resulting curve traced by the reflector is called a curtate cycloid. One such cycloid appears in the next figure, where a = 3 and b = 2.



   Using vector methods, find a set of parametric equations for the **curtate cycloid**.

1.3 Section 1.3

1. Find three nonparallel unit vectors that are perpendicular to $i - j + k$.

2. Calculate the angle between:

   (a) $\mathbf{a} = (\sqrt{3}, 1)$, $\mathbf{b} = (-\sqrt{3}, 1)$.
   (b) $\mathbf{a} = (-1, 2)$, $\mathbf{b} = (3, 1)$.
   (c) $\mathbf{a} = (1, -2, 3)$, $\mathbf{b} = (3, -6, -5)$.

3. Calculte the $proj_a \mathbf{b}$ of:

(a) $\mathbf{a} = (1,1,0)$, $\mathbf{b} = (2,3,-1)$.

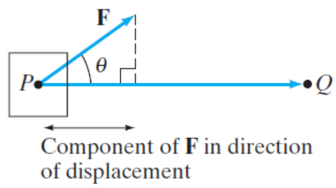
(b) $\mathbf{a} = (0,0,5)$, $\mathbf{b} = (1,-1,2)$.

(c) $\mathbf{a} = (1,1,2)$, $\mathbf{b} = (2,-4,1)$. Do it again but use the generic vector $\tilde{\mathbf{a}} = k \cdot \mathbf{a}$.

4. In physics, when a constant force acts on an object as the object is displaced, the **work** done by the force is the product of the length of the displacement and the component of the force in the direction of the displacement. Below figure depicts an object acted upon by a constant force $\mathbf{F}$, which displaces it from the point P to the point Q. Let θ denote the angle between F and the direction of displacement.

   (a) Show that the work done by $\mathbf{F}$ is determined by the formula $\mathbf{F} \cdot \overrightarrow{PQ}$.

   (b) Find the work done by the (constant) force $\mathbf{F} = i + 5j + 2k$ in moving a particle from the point $(1,-1,1)$ to the point $(2,0,-1)$.



5. Suppose that a force $F = i - 2j$ is acting on an object moving parallel to the vector a = 4i + j. Decompose F into a sum of vectors F_1 and F_2, where F_1 points along the direction of motion (hint: use projection) and F_2 is perpendicular to the direction of motion (hint: use subtraction).

6. Using vectors, prove that the diagonals of a parallelogram are perpendicular if and only if the parallelogram is a rhombus. (Note: A **rhombus** is a parallelogram whose four sides all have the same length.)

1.4 Section 1.4

1. Evaluate the determinants:

1. $\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix}$ **2.** $\begin{vmatrix} 0 & 5 \\ -1 & 6 \end{vmatrix}$

3. $\begin{vmatrix} 1 & 3 & 5 \\ 0 & 2 & 7 \\ -1 & 0 & 3 \end{vmatrix}$ **4.** $\begin{vmatrix} -2 & 0 & \frac{1}{2} \\ 3 & 6 & -1 \\ 4 & -8 & 2 \end{vmatrix}$

2. If $\mathbf{a} \times \mathbf{b} = (3,-7,-2)$, what is $(\mathbf{a}+\mathbf{b}) \times (\mathbf{a}-\mathbf{b})$?

3. Calculate the area of the parallelogram having vertices $(1,1)$, $(3,2)$, $(1,3)$, and $(-1,2)$.

4. Find a unit vector that is perpendicular to both $(2,1,-3)$ and $(1,0,1)$.

5. Suppose that you are given nonzero vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{R}^3$. Use dot and cross products to give expressions for vectors satisfying the following geometric descriptions:

   (a) A vector orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

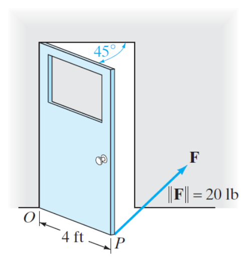
   (b) A vector of length 2 orthogonal to $\mathbf{a}$ and $\mathbf{b}$.

   (c) The vector projection of $\mathbf{b}$ onto $\mathbf{a}$.

   (d) A vector with the length of $\mathbf{b}$ and the direction of $\mathbf{a}$.

   (e) A vector orthogonal to $\mathbf{a}$ and $\mathbf{b} \times \mathbf{c}$.

6. Egbert applies a 20 lb force at the edge of a 4 ft wide door that is half-open in order to close it (See Figure). Assume that the direction of force is perpendicular to the plane of the doorway. What is the torque about the hinge on the door?



1.5 Section 1.5

1. Find an equation for the plane containing the points $(A, 0, 0)$, $(0, B, 0)$, and $(0, 0, C)$. Assume that at least two of A, B, and C are nonzero.

2. Find a single equation of the form $Ax + By + Cz = D$ that describes the plane given parametrically as $x = 3s - t + 2$, $y = 4s + t$, $z = s + 5t + 3$. (Hint: Begin by writing the parametric equations in vector form and then find a vector normal to the plane.)

3. Determine the distance between the two lines $l_1(t) = t(8, -1, 0) + (-1, 3, 5)$ and $l_2(t) = t(0, 3, 1) + (0, 3, 4)$.

4. Find the distance between the point $(-11, 10, 20)$ and the line $l : x = 5 - t$, $y = 3$, $z = 7t + 8$.

5. Find the distance between the two planes given by the equations $x - 3y + 2z = 1$ and $x - 3y + 2z = 8$.

6. Two planes are given parametrically by the vector equations

$$x_1(s, t) = (-3, 4, -9) + s(9, -5, 9) + t(3, -2, 3)$$
$$x_2(s, t) = (5, 0, 3) + s(-9, 2, -9) + t(-4, 7, -4)$$

   (a) Give a convincing explanation for why these planes are parallel.

   (b) Find the distance between the planes.

7. Suppose that $l_1(t) = ta + b_1$ and $l_2(t) = ta + b_2$ are parallel lines in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Show that the distance D between them is given by

$$D = \frac{\|\mathbf{a} \times (\mathbf{b_2} - \mathbf{b_1})\|}{\|\mathbf{a}\|}$$

1.6 Section 1.6

1. Calculate the following, where $\mathbf{a} = (1, 3, 5, ..., 2n-1)$ and $\mathbf{b} = (2, -4, 6, ..., (-1)^{(n+1)}2n)$:

   (a) $\mathbf{a} + \mathbf{b}$

   (b) $\mathbf{a} - \mathbf{b}$

   (c) $\mathbf{a} \cdot \mathbf{b}$

   (d) $\|\mathbf{a}\|$

2. Verify the Pythagorean theorem. That is, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors in $\mathbb{R}^n$ such that $\mathbf{a} + \mathbf{b} = \mathbf{c}$ and $\mathbf{a} \cdot \mathbf{b} = 0$, then
$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = \|\mathbf{c}\|^2.$$

3. Suppose that you run a grain farm that produces six types of grain at prices of \$200, \$250, \$300, \$375, \$450, \$500 per ton.

   (a) If $x = (x_1, ..., x_6)$ is the commodity bundle vector (meaning that x_i is the number of tons of grain i to be purchased), express the total cost of the commodity bundle as a dot product of two vectors in $\mathbb{R}^6$.

   (b) A customer has a budget of \$100,000 to be used to purchase your grain. Express the **set** of possible commodity bundle vectors that the customer can afford. Also describe the relevant budget hyperplane in $\mathbb{R}^6$.

4. Calculate the indicated matrix quantities:

   (a) $3A - 2B$

   (b) DB

   (c) AC

   (d) $B^t D$

   where:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -4 & 9 & 5 \\ 0 & 3 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & 7 \\ 0 & 3 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}.$$

5. Compute:

$$|M| = \begin{vmatrix} 7 & 0 & -1 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & -3 & 0 & 2 \\ 0 & 5 & 1 & -2 \end{vmatrix}$$

6. An **upper triangular** matrix is an $n \times n$ matrix whose entries below the **main diagonal** are all zero. (Note: The main diagonal is the diagonal going from upper left to lower right.) For example, the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & 3 & 4 & 3 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

is upper triangular.

   (a) Give an analogous definition for a **lower triangular** matrix and also an example of one.

   (b) Use *minors* expansion to show that the determinant of any n × n upper or lower triangular matrix A is the product of the entries on the main diagonal.

1.7 Section 1.7

1. Find the Cartesian coordinates of the points whose polar coordinates are given:

   (a) $(\sqrt{2}, \frac{\pi}{4})$

   (b) $(\sqrt{3}, 5\frac{\pi}{6})$

   (c) $(3, 0)$

2. Find the polar coordinates of the points whose Cartesian coordinates are given:

   (a) $(2\sqrt{3}, 2)$

   (b) $(-2, 2)$

   (c) $(-1, -2)$

3. Find the Cartesian coordinates of the points whose cylindrical coordinates are given:

   (a) $(2, 2, 2)$

   (b) $(\pi, \frac{\pi}{2}, 1)$

   (c) $(1, 2\frac{\pi}{3}, -2)$

4. Find the rectangular coordinates of the points whose spherical coordinates are given (ρ, φ, θ):

   (a) $(4, \frac{\pi}{2}, \frac{\pi}{3})$

   (b) $(3, \frac{\pi}{3}, \frac{\pi}{2})$

   (c) $(1, \frac{3\pi}{4}, \frac{2\pi}{3})$

5. Find the spherical coordinates of the points whose rectangular coordinates are given:

   (a) $(1, -1, \sqrt{6})$

   (b) $(0, \sqrt{3}, 1)$

6. This problem concerns the surface described by the equation $(r-2)^2 + z^2 = 1$ in cylindrical coordinates. (Assume $r \geq 0$.)

(a) Sketch the intersection of this surface with the halfplane $\theta = \pi/2$.

(b) Sketch the entire surface.

7. Translate the following equations from the given coordinate system into equations in each of the other two systems and try to sketch it.

(a) $\rho \sin\varphi \sin\theta = 2$.

(b) $z^2 = 2x^2 + 2y^2$

(c) $r = 0$

8. Consider the solid in $\mathbb{R}^3$ shown in Figure below. Describe it, using spherical coordinates.