Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 4.1 Path Integrals

Problem 1. Compute the following path integrals:

i) $f(x, y) = 2xy^2$ along the circle of radius R in the first quadrant.

ii) $f(x, y, z) = (x^2 + y^2 + z^2)^2$ along the helix $\mathbf{r}(t) = (\cos t, \sin t, 3t)$, from the point $(1, 0, 0)$ to the point $(1, 0, 6\pi)$.

Solution: $i)$ $2R^4/3$; $ii)$ $2\pi\sqrt{10}(5 + 120\pi^2 + 1296\pi^4)/5$.

Problem 2. Compute the path integrals of the vector field $\mathbf{F}$ along the given paths:

i) $\mathbf{F}(x, y) = (x^2 - 2xy, y^2 - 2xy)$, along the parabola $y = x^2$ from $(-1, 1)$ to $(1, 1)$,

ii) $\mathbf{F}(x, y) = (x^2 + y^2, x^2 - y^2)$, along the curve $y = 1 - |1 - x|$, from $(0, 0)$ to $(2, 0)$,

iii) $\mathbf{F}(x, y, z) = (y^2 - z^2, 2yz, -x^2)$, along the path given by $r(t) = (t, t^2, t^3)$, for $t \in [0, 1]$,

iv) $\mathbf{F}(x, y, z) = (2xy, x^2 + z, y)$, along the line that connects $(1, 0, 2)$ with $(3, 4, 1)$.

Solution: $i)$ $-14/15$; $ii)$ $4/3$; $iii)$ $1/35$; $iv)$ 40.

Problem 3. Consider the vector function $\mathbf{F}(x, y) = (x^2, y)$. Compute the path integral of $\mathbf{F}$ along the following paths that start at $(1, 0)$ and end at $(-1, 0)$:

i) The line segment connecting both points.

ii) The two possible paths of the rectangle $[-1, 1] \times [-1, 1]$.

iii) The upper semi-circle that connects both points.

Solution: $-2/3$ in all cases.

Problem 4. Compute:

i) $\int_g (x - y)dx + (x + y)dy$, where g is the line connecting $(1, 0)$ with $(0, 2)$.

ii) $\int_C x^3 dy - y^3 dx$, where C is the unit circle.

iii) $\displaystyle\int_\Gamma \frac{dx + dy}{|x| + |y|}$, where Γ is the square of vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$, walked on once in a counterclockwise direction.

iv) $\int_\rho (x + 2y)dx + (3x - y)dy$ where ρ is the ellipse defined by $x^2 + 4y^2 = 4$, walked on once in a counterclockwise direction.

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v) $\int_R \frac{y^3dx - xy^2dy}{x^5}$, where R is the curve $x = \sqrt{1-t^2}$, $y = t\sqrt{1-t^2}$, $-1 \le t \le 1$.

Solution: $i)$ $7/2$; $ii)$ $3\pi/2$; $iii)$ 0; $iv)$ 2π; $v)$ $-\pi/2$.

Problem 5. Compute:

i) $\int_\gamma y\,dx - x\,dy + z\,dz$, where γ is the curve resulting from the intersection of the cylinder $x^2+y^2 = a^2$ and the plane $z - y = a$ and oriented counterclockwise.

ii) $\int_\gamma \mathbf{F}$, where $\mathbf{F}(x,y,z) = (2xy + z^2, x^2, 2xz)$ and γ is the curve resulting from the intersection of the plane $x = y$ and the sphere $x^2 + y^2 + z^2 = a^2$, oriented positively.

iii) $\int_\gamma \mathbf{F}$, where $\mathbf{F}(x,y,z) = (y,z,x)$ and γ is the intersection of $x^2 + y^2 = 2x$ with $x = z$.

Solution: $i)$ $-2\pi a^2$; $ii)$ 0; $iii)$ 0.

Problem 6. A particle of mass m moves along the curve

$$\mathbf{r}(t) = (t^2, \sin t, \cos t),\ t \in [0,1].$$

Assuming Newton's second law $\mathbf{F}(t) = m\mathbf{r}''(t)$, compute the force that acts on the particle. Compute also the total work done by this force field.

Solution: $\mathbf{F}(t) = m(2, -\sin t, -\cos t)$ and the work done by the force field along the curve is $2m$.

Problem 7. Find the value of $b > 0$ that minimises the work done by the force field $\mathbf{F}(x,y) = (3y^2+2, 16x)$ for moving a particle from $(-1,0)$ to $(1,0)$ along the semi-ellipse $b^2x^2 + y^2 = b^2$, $y \ge 0$.

Solution: The minimal work done is $8(2-\pi)/3$, for $b = 4/3$.

Problem 8. Consider the force field $\mathbf{F}(x,y) = (cxy, x^6y^2)$, $a,b,c > 0$. Compute the value of a as a function of c in order that the work done by this force field in moving a particle along the parabola $y = ax^b$ from $x = 0$ to $x = 1$ does not depend on b.

Solution: $a = \sqrt{3c/2}$.

Problem 9. Compute the work done by the force field $\mathbf{F}(r,\theta) = (-4\sin\theta, 4\sin\theta)$ (given in polar coordinates) while moving a particle along the curve $r = e^{-\theta}$ from $(1,0)$ to the origin.

Solution: $8/5$.