The Finite Element Method

Section 7

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Section Objectives

We have now seen how to carry out a finite element approximation of a 2-D BVP with a scalar field variable (and we will later extend this for vector field variables). In addition we have shown how compatible elements may be generated which have shape functions that satisfy certain fundamental requirements.

In this section we will now introduce some additional techniques which facilitate the implementation of the finite element method, specifically:

- the use of numerical integration for finite elements for which the required functions are cumbersome or do not have an analytical solution
- the change of variables in shape function derivatives by means of the Jacobean determinant
- the concise formulation of elemental 'stiffness' matrices

Numerical Integration

We are aiming to compute the integral

$$\int_{\Omega^e} f(\mathbf{x}) d\Omega$$

where, for example, $f = \mathbf{B}_a^T \mathbf{D} \mathbf{B}_b$ with $B_a = \nabla N_a$. This integration may be carried out in the element's parent domain by a change of variables, i.e.

$$n_{sd} = 1 \qquad \int_{\Omega^e} f(x) dx = \int_{-1}^{1} f(x(\xi)) x_{,\xi}(\xi) d\xi$$

$$n_{sd} = 2 \qquad \int_{\Omega^e} f(x,y) d\Omega = \int_{-1}^{1} \int_{-1}^{1} f(x(\xi,\eta), y(\xi,\eta)) j(\xi,\eta) d\xi d\eta$$

1-D Numerical Integration

We first consider the numerical integration of a function $g(\xi)$ when $n_{sd} = 1$. The integration is approximated by a summation

$$\int_{-1}^{1} g(\xi) d\xi = \sum_{l=1}^{n_{int}} g(\tilde{\xi}_l) w_l + R \approx \sum_{l=1}^{n_{int}} g(\tilde{\xi}_l) w_l$$

where n_{int} is the number of integration points.

Possible approaches to choosing ξ_l and w_l are the trapezium rule, or for greater accuracy, Simpson's rule: $n_{int} = 3$, $\tilde{\xi}_1 = -1$, $\tilde{\xi}_2 = 0$, $\tilde{\xi}_3 = 1$, $w_1 = w_3 = 1/3$, and $w_2 = 4/3$. In this case the error is

$$R = \frac{-g^{(4)}(\xi)}{90}$$

where $g^{(4)} = g_{,\xi\xi\xi\xi}$.

Numerical integration by Simpson's rule is therefore 4th-order accurate.

Gaussian Quadrature

An alternative numerical integration technique is Gaussian quadrature. In this case, if $n_{int} = 1$

$$\tilde{\xi}_1 = 0, \; w_1 = 2, \; R = \frac{g_{,\xi\xi}(\tilde{\xi})}{3}$$

i.e. the solution is 2nd-order accurate. If $n_{int} = 2$

$$\tilde{\xi}_1 = -\frac{1}{\sqrt{3}}, \; \tilde{\xi}_2 = \frac{1}{\sqrt{3}}, \; w_1 = w_2 = 1, \; R = \frac{g^{(4)}(\tilde{\xi})}{135}$$

i.e. the solution is 4th-order accurate.

If we compare this with integration by Simpson's rule, we can see that the same accuracy of integration is achieved with one fewer integration points. This results in significant computational savings.

In multiple dimensions, 1-D quadrature rules are applied on each coordinate separately.

Shape Function Derivatives

Recall the definition of the $\mathbf{B}_a$ matrix which is used to evaluate the element stiffness matrix requires the differentiation of the shape functions to obtain (in two dimensions, i.e. $n_{sd} = 2$) $N_{a,x}$ and $N_{a,y}$. These may be expanded using the chain rule of differentiation to obtain

$$\begin{aligned} N_{a,x} &= N_{a,\xi}\xi_{,x} + N_{a,\eta}\eta_{,x} \\ N_{a,y} &= N_{a,\xi}\xi_{,y} + N_{a,\eta}\eta_{,y} \end{aligned}$$

We cannot obtain these directly as we do not know $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. We do, however, know

$$x(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta) x_a^e$$

$$y(\xi, \eta) = \sum_{a=1}^{n_{en}} N_a(\xi, \eta) y_a^e$$

Jacobean Determinant

Hence we can calculate

$$\mathbf{x}_{,\xi} = \begin{bmatrix} x_{,\xi} & x_{,\eta} \\ y_{,\xi} & y_{,\eta} \end{bmatrix}$$

in which

$$x_{,\xi} = \sum_{a=1}^{n_{en}} N_{a,\xi} x_a^e$$

etc.

We can then obtain the terms we require to differentiate the shape functions by inverting the above expression for $\mathbf{x}_{,\xi}$.

$$(\mathbf{x}_{,\xi})^{-1} = \begin{bmatrix} \xi_{,x} & \xi_{,y} \\ \eta_{,x} & \eta_{,y} \end{bmatrix} = \frac{1}{j} \begin{bmatrix} y_{,\eta} & -x_{,\eta} \\ -y_{,\xi} & x_{,\xi} \end{bmatrix}$$

in which j is the Jacobean defined as

$$j = det(\mathbf{x}_{,\xi}) = x_{,\xi} y_{,\eta} - x_{,\eta} y_{,\xi}$$

Element Stiffness Formulation

It is also convenient and efficient to evaluate the element stiffness matrix with respect to the parent domain.

$$\mathbf{k}^e = \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} d\Omega = \int_{\square} \mathbf{B}^T \mathbf{D} \mathbf{B} j d\square$$

Carrying out this integration using quadrature gives

$$\mathbf{k}^e \approx \sum_{l=1}^{n_{int}} (\mathbf{B}^T \mathbf{D} \mathbf{B} j)\Big|_{\xi_l} w_l$$

If we define

$$\tilde{\mathbf{D}} = j(\xi_l) w_l \mathbf{D}$$

we can write

$$\mathbf{k}^e \approx \sum_{l=1}^{n_{int}} (\mathbf{B}^T \tilde{\mathbf{D}} \mathbf{B})_l$$

Summary by Example

We have introduced two mathematical techniques that facilitate the implementation of the Finite Element Method:

- Numerical integration
- Generation of shape function derivatives

The techniques that have been introduced in this section are best illustrated by example

Visualizer

Demonstration of the generation of an elemental stiffness matrix in global coordinates using Gaussian quadrature for a 4-node quadrilateral element