UNIT 2

Exercises to UNIT 2- Part I: Temporal Characteristics

2.1 A two-level random binary process is defined by

$X(t) = A$ or $-A, (n-1)T < t < nT$ where the levels A and $-A$ occur with equal probability, T is a positive constant, and $n = 0, \pm 1, \pm 2, \ldots$

(a) Sketch a typical sample function

(b) Classify the process

2.2 A random process is defined by $X(t) = A$, where A is a continuous random variable uniformly distributed on $[0, 1]$.

(a) Determine the form of the sample functions

(b) Classify the process

2.3 Given the random process $X(t) = Acos(\omega_0 t + \Theta)$ where A and ω_0 are constants and Θ is a uniformly distributed random variable on the interval $[0, 2\pi]$. Find

(a) The mean

(b) The autocorrelation

(c) The time average

(d) The time autocorrelation

2.4 Given the random process

$X(t) = Asin(\omega_0 t + \Theta)$

where A and ω_0 are constants and Θ is a uniformly distributed random variable on the interval $[-\pi, \pi]$. Define a new random variable $Y(t) = X^2(t)$.

(a) Find the autocorrelation function of $Y(t)$

(b) Find the cross-correlation function of $X(t)$ and $Y(t)$

(c) Are $X(t)$ and $Y(t)$ wide-sense stationary?

(d) Are $X(t)$ and $Y(t)$ jointly wide-sense stationary?

2.5 Given the random process

$Y(t) = X(t)cos(\omega_0 t + \Theta)$

where $X(t)$ is a wide-sense stationary random process that amplitude-modulates a carrier of constant angular frequency ω_0 with a random phase Θ independent of $X(t)$ and uniformly distributed on $[-\pi, \pi]$

(a) Find $E[Y(t)]$

(b) Find the autocorrelation of $Y(t)$

(c) Is $Y(t)$ wide-sense stationary?

2.6 The random processes $X(t)$ and $Y(t)$ are statistically independent, have zero mean and have autocorrelation functions

$$\begin{aligned} R_{XX}(\tau) &= e^{-|\tau|} \\ R_{YY}(\tau) &= cos(2\pi\tau) \end{aligned}$$

(a) Find the autocorrelation function of $W_1(t) = X(t) + Y(t)$

(b) Find the autocorrelation function of $W_2(t) = X(t) - Y(t)$

(c) Find the cross-correlation function of $W_1(t)$ and $W_2(t)$

2.7 Given two w.s.s. random processes $X(t)$ and $Y(t)$. Find expressions for the autocorrelation function of $W(t) = X(t) + Y(t)$ if:

(a) $X(t)$ and $Y(t)$ are correlated

(b) $X(t)$ and $Y(t)$ are uncorrelated

(c) $X(t)$ and $Y(t)$ are uncorrelated with zero mean

Try to use means over cross-correlations if possible.

2.8 Consider random processes

$$Y_1(t) = X(t)cos(\omega_0 t)$$
$$Y_2(t) = Y(t)cos(\omega_0 t + 2\Theta)$$

where $X(t)$ and $Y(t)$ are jointly wide-sense stationary processes.

(a) If $\Theta \sim U(\theta_0, \theta_1)$ and independent of $X(t)$ and $Y(t)$, are there any conditions on Θ that will make $Y_1(t)$ and $Y_2(t)$ orthogonal?

2.9

Consider random processes

$$X(t) = Acos(\omega_0 t + \Theta)$$
$$Y(t) = Bsin(\omega_0 t + \Theta)$$

where A, B, and ω_0 are constants while Θ is a random variable uniform on $[0, 2\pi]$. $X(t)$ and $Y(t)$ are zero-mean, wide-sense stationary with autocorrelation functions:

$$R_{XX}(\tau) = (A^2/2)cos(\omega_0 \tau)$$
$$R_{YY}(\tau) = (B^2/2)cos(\omega_0 \tau)$$

Are $X(t)$ and $Y(t)$ jointly wide-sense stationary?

Exercises to UNIT 2- Part II: Spectral Characteristics

2.10

Given that $X(t) = \sum_{i=1}^{N} \alpha_i X_i(t)$ where α_i are real constants, show that

$\mathcal{S}_{XX}(\omega) = \sum_{i=1}^{N} \alpha_i^2 \mathcal{S}_{X_iX_i}(\omega)$

if

(a) the processes $X_i(t)$ are orthogonal

(b) the processes are independent with zero mean

2.11

If $X(t)$ is a stationary process, find the power spectrum of

$Y(t) = A_0 + B_0 X(t)$

in terms of the power spectrum of $X(t)$ if A_0 and B_0 are real constants.

2.12 The autocorrelation function of a random process $X(t)$ is

$R_{XX}(\tau) = 3 + 2e^{-4\tau^2}$

Find

(a) The power spectrum of $X(t)$

(b) The average power of $X(t)$

(c) The fraction of power that lies in the frequency band $-1/\sqrt{2} \leq \omega \leq 1/\sqrt{2}$

2.13 Given a random process with autocorrelation $R_{XX}(\tau) = P cos^4(\omega_0 \tau)$, find

(a) $\mathcal{S}_{XX}(\omega)$

(b) P_{XX} from $\mathcal{S}_{XX}(\omega)$

(c) P_{XX} from $R_{XX}(\tau)$

2.14 Given a random process with autocorrelation

$R_{XX}(\tau) = Ae^{-\alpha|\tau|} cos(\omega_0 \tau)$

where $A > 0$, $\alpha > 0$, and ω_0 are real constants, find the power spectrum.

2.15 A random process is given by

$W(t) = AX(t) + BY(t)$

where A and B are real constants and $X(t)$ and $Y(t)$ are jointly wide-sense stationary processes. Find

(a) The power spectrum of $W(t)$ as a function of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $S_{XY}(\omega)$ and $S_{YX}(\omega)$

(b) The power spectrum of $W(t)$ as a function of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $\overline{X}$ and $\overline{Y}$, if $X(t)$ and $Y(t)$ are uncorrelated

(c) $\mathcal{S}_{XW}(\omega)$ and $\mathcal{S}_{WX}$ as functions of $S_{XX}(\omega)$, $S_{YY}(\omega)$, $S_{XY}(\omega)$ and $S_{YX}(\omega)$

2.16 A wide-sense stationary $X(t)$ is applied to an ideal differentiator, so that $Y(t) = dX(t)/dt$. The cross-correlation of the input-output processes is known to be

$R_{XY}(\tau) = dR_{XX}(\tau)/dt$

(a) Determine $\mathcal{S}_{XY}(\omega)$ in terms of $\mathcal{S}_{XX}(\omega)$

(b) Determine $\mathcal{S}_{YX}(\omega)$ in terms of $\mathcal{S}_{XX}(\omega)$

2.17 The cross-correlation of jointly wide-sense stationary processes $X(t)$ and $Y(t)$ is assumed to be

$R_{XY}(\tau) = Be^{-W\tau}u(\tau)$

where $B > 0$ and $W > 0$ are constants. Find

(a) $R_{YX}(\tau)$

(b) $\mathcal{S}_{XY}(\omega)$ (use appendix C from Peebles' book)

(c) $\mathcal{S}_{YX}(\omega)$ (use cross-power density properties)

2.18 Consider two random process $X_1(t)$ and $X_2(t)$. The mean of $X_1(t)$ is equal to A ($A > 0$) and $X_2(t)$ is a white noise with power density 5 W/(rad/s). Given an LTI system with impulse response

$h(t) = e^{-\alpha t}u(t)$.

with $\alpha > 0$. Find

(a) The mean value of the response of the LTI system if the input is $X_1(t)$

(b) The average power (second-order moment) of the response of the system if the input is $X_2(t)$.

2.19 A random process $X(t)$ with known mean $\bar{X}$ is the input of an LTI system with impulse response

$h(t) = te^{-Wt}u(t)$.

Find

(a) The mean value of the response of the LTI system

(b) The average power (second-order moment) of the response of the system if $X(t)$ is a white noise with power density 5 W/(rad/s).

2.20 A white noise with power density $N_0/2$ is applied to a network with impulse response of a system with impulse response

$h(t) = Wte^{-Wt}u(t)$

where W is a real positive constant. Find the cross-correlation of the reponse of input and the output of the system.

2.21 A stationary random process $X(t)$, having an autocorrelation function $R_{XX} = 2e^{-4|\tau|}$ is applied to the network of the figure below. Find the power spectrum of the output of the system.