Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 10

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Outline

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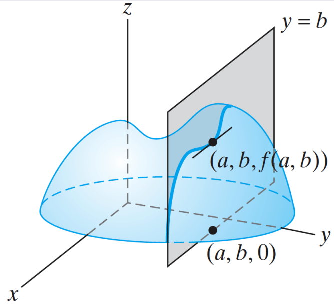
Recalling Geometric Interpretation of Partial Derivatives

$$\frac{\partial f}{\partial x}(a, b)$$

Geometrically it is the slope at the point $(a, b, f(a, b))$ of the curve obtained by intersecting:

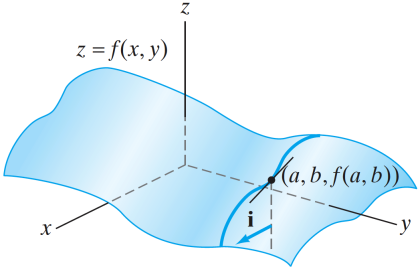
- The surface $z = f(x, y)$ with the plane $y = b$.



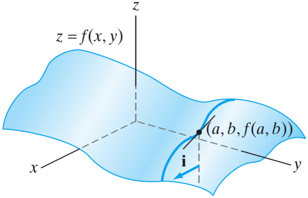
Partial Derivatives as Directional Derivatives

Consider an alternative geometric way to view $\frac{\partial f}{\partial x}(a, b)$:

- It can be viewed as the rate of change of f as we move "infinitesimally" from $\mathbf{a} = (a, b)$ in the $\mathbf{i}$-direction.



Partial Derivatives as Directional Derivatives



- By the definition of the partial derivative:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h} = \lim_{h\to 0}\frac{f((a,b)+(h,0))-f(a,b)}{h}$$
$$= \lim_{h\to 0}\frac{f((a,b)+h(1,0))-f(a,b)}{h} = \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{i})-f(\mathbf{a})}{h}$$

- Similarly, we have,

$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0}\frac{f(\mathbf{a}+h\mathbf{j})-f(\mathbf{a})}{h}$$

General Directional Derivatives

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{i}) - f(\mathbf{a})}{h}$$
$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{j}) - f(\mathbf{a})}{h}$$

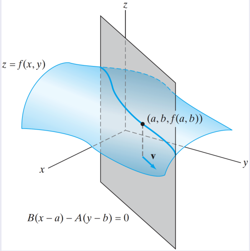
- Partial derivatives are special cases of a more general type of derivative,
- Suppose $\mathbf{v}$ is any unit vector in $\mathbb{R}^2$ and consider the quantity,

$$\lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{v}) - f(\mathbf{a})}{h}$$

It is the rate of change of f as we move (infinitesimally) from $\mathbf{a} = (a,b)$ in the direction specified by $\mathbf{v} = (A,B) = A\mathbf{i} + B\mathbf{j}$

General Directional Derivatives

$$\lim_{h\to 0} \frac{f(\mathbf{a}+h\mathbf{v})-f(\mathbf{a})}{h}, \quad \mathbf{v}=(A,B)=A\mathbf{i}+B\mathbf{j}$$



- It is also the slope of the curve obtained as the intersection of
  - The surface $z = f(x, y)$, with the vertical plane,

$$B(x-a)-A(y-b)=0$$

Outline

Definition 6.1

- Let X be an open in $\mathbb{R}^n$ and $\mathbf{a} \in X$.
- Let $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be a scalar-valued function.
- *If* $\mathbf{v} \in \mathbb{R}^n$ *is any unit vector, then the directional derivative of* f *at* $\mathbf{a}$ *in the direction of* $\mathbf{v}$ *is:*

$$D_{\mathbf{v}} f(\mathbf{a}) = \lim_{h \to 0} \frac{f(\mathbf{a} + h\mathbf{v}) - f(\mathbf{a})}{h}$$

Provided that
this limit exists.

Example 1

- Suppose,

$$f(x,y) = x^2 - 3xy + 2x - 5y$$

- Then, if $\mathbf{v} = (v, w) \in \mathbb{R}^2$ is any unit vector, it follows that

$$D_{\mathbf{v}} f(0,0) = \lim_{h \to 0} \frac{f((0,0) + h(v,w)) - f(0,0)}{h}$$
$$= \lim_{h \to 0} \frac{h^2 v^2 - 3h^2 vw + 2hv - 5hw}{h} = \lim_{h \to 0} (hv^2 - 3hvw + 2v - 5w)$$
$$= 2v - 5w$$

- Thus
  - The rate of change of f is $2v - 5w$ if we move from the origin in the direction given by $\mathbf{v}$.
  - The rate of change is zero if $\mathbf{v} = (5/\sqrt{29}, 2/\sqrt{29})$ or $\mathbf{v} = (-5/\sqrt{29}, -2/\sqrt{29})$.

Theorem 6.2

- Let X be an open in $\mathbb{R}^n$
- Suppose $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in X$
- Then the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$ exists for all directions (unit vectors) $\mathbf{v} \in \mathbb{R}^n$ and, moreover, we have

$$D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Example 2

- Recall function in Example 1

$$f(x, y) = x^2 - 3xy + 2x - 5y$$

- For any unit vector $\mathbf{v} = v\mathbf{i} + w\mathbf{j} \in \mathbb{R}^2$

$$\begin{aligned} D_{\mathbf{v}} f(0,0) &= \nabla f(0,0) \cdot \mathbf{v} = (f_x(0,0)\mathbf{i} + f_y(0,0)\mathbf{j}) \cdot (v\mathbf{i} + w\mathbf{j}) \\ &= (2\mathbf{i} - 5\mathbf{j}) \cdot (v\mathbf{i} + w\mathbf{j}) = 2v - 5w \end{aligned}$$

Theorem 6.2

- Let X be an open in $\mathbb{R}^n$
- Suppose $\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable at $\mathbf{a} \in X$
- Then the directional derivative $D_{\mathbf{v}} f(\mathbf{a})$ exists for all directions (unit vectors) $\mathbf{v} \in \mathbb{R}^n$ and, moreover, we have

$$D_{\mathbf{v}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{v}$$

Remark

- The converse of Theorem 6.2 does not hold

A function may have directional derivatives
in all directions at a point yet fail to be differentiable

Theorem 6.3: Steepest Ascent

The directional derivative $D_{\mathbf{u}}f(\mathbf{a})$ is:

- Maximized, with respect to direction, when $\mathbf{u}$ points in the same direction as $\nabla f(\mathbf{a})$, and
- Minimized, when $\mathbf{u}$ points in the opposite direction.

Furthermore, the maximum and minimum values of $D_{\mathbf{u}}f(\mathbf{a})$ are:

$$\|\nabla f(\mathbf{a})\| \quad \text{and} \quad -\|\nabla f(\mathbf{a})\|, \quad \text{respectively}$$

Example 4

- Imagine you are traveling in space near the planet Nilrebo.
- Suppose that one of your spaceship's instruments measures the external atmospheric pressure on your ship .
- This external atmospheric pressure is measured as a function $f(x, y, z)$ of position.
- Assume that this function f is differentiable.
- Then Theorem 6.2 can be applied.
- If you travel from point $\mathbf{a} = (a, b, c)$ in the direction of the (unit) vector $\mathbf{u} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$, the rate of change of pressure is given by

$$D_{\mathbf{u}} f(\mathbf{a}) = \nabla f(\mathbf{a}) \cdot \mathbf{u}$$

In what direction
is the pressure increasing the most?

Example 4

- In particular, suppose the pressure function on Nilrebo is

$$f(x, y, z) = 5x^2 + 7y^4 + x^2z^2 \text{ atm}$$

- Assume the origin is located at the center of Nilrebo and distance units are measured in thousands of kilometers
- The rate of change of pressure at $(1, -1, 2)$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$ may be calculated as

$$\nabla f(1, -1, 2) \cdot \mathbf{u}, \quad \text{where } \mathbf{u} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3} \ \text{(normalized)}$$

- Using Theorem 6.2

$$\begin{aligned} D_{\mathbf{u}} f(1, -1, 2) &= \nabla f(1, -1, 2) \cdot \mathbf{u} = 18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k} \cdot \frac{(\mathbf{i} + \mathbf{j} + \mathbf{k})}{\sqrt{3}} \\ &= \frac{18 - 28 + 4}{\sqrt{3}} = -2\sqrt{3} \text{ atm/Mm} \end{aligned}$$

Example 4

- Suppose the pressure function on Nilrebo is

$$f(x, y, z) = 5x^2 + 7y^4 + x^2z^2 \text{ atm}$$

- Assume the origin is located at the center of Nilrebo and distance units are measured in thousands of kilometers
- The rate of change of pressure at $(1, -1, 2)$ in the direction of $\mathbf{i} + \mathbf{j} + \mathbf{k}$ may be calculated as

  $\nabla f(1, -1, 2) \cdot \mathbf{u}$, where $\mathbf{u} = (\mathbf{i} + \mathbf{j} + \mathbf{k})/\sqrt{3}$ (normalized)

- Using Theorem 6.3, the pressure will increase most rapidly in the direction of $\nabla f(1, -1, 2)$, that is, in the direction

$$\frac{18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k}}{\|18\mathbf{i} - 28\mathbf{j} + 4\mathbf{k}\|} = \frac{9\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}}{\sqrt{281}}$$

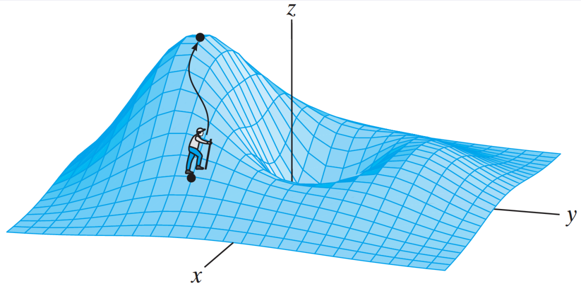
- The rate of this increase is $\|\nabla f(1, -1, 2)\| = 2\sqrt{281}$ atm/Mm

Steepest Ascent

- Theorem 6.3 is independent of the dimension

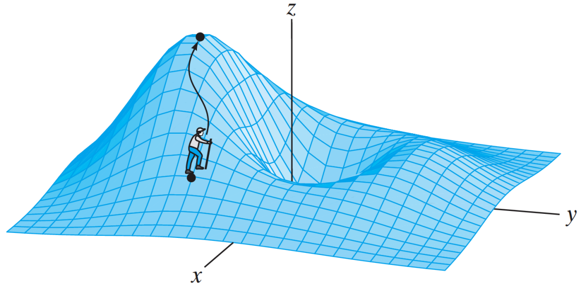
It applies to functions
$\mathbf{f} : X \subseteq \mathbb{R}^n \to \mathbb{R}$ for any $n \geq 2$

- For $n = 2$, there is another geometric interpretation of Theorem 6.3
- Suppose you are mountain climbing on the surface $z = f(x, y)$



Steepest Ascent

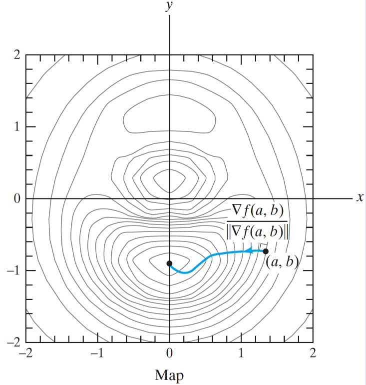
- Suppose you are mountain climbing on the surface $z = f(x, y)$



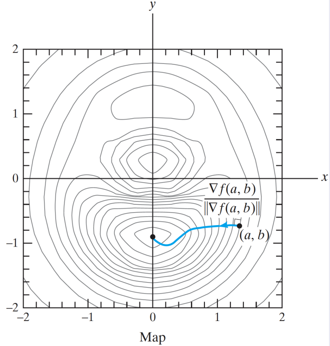
- Think of the value of f as the height of the mountain above (or below) sea level

Steepest Ascent

- Suppose you are equipped with a map and compass, which supply information in the xy-plane only

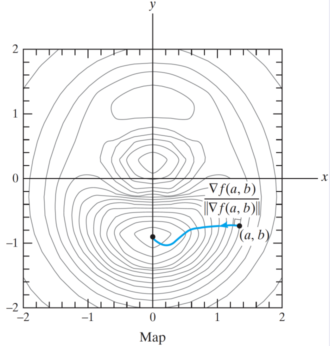


Steepest Ascent



- Assume you are at the point on the mountain with xy-coordinates (map coordinates) (a, b)
- To climb the mountain faster, Theorem 6.3 says that you should move in the direction parallel to the gradient $\nabla f(a, b)$

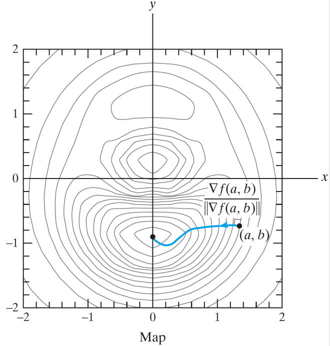
Steepest Ascent



- Similarly, you should move in the direction parallel to $-\nabla f(a, b)$ in order to descend most rapidly
- Moreover, the slope of your ascent or descent in these cases is

$$\|\nabla f(a, b)\|$$

Steepest Ascent



$\nabla f(a, b)$ is a vector in $\mathbb{R}^2$ that gives
the optimal north-south, east-west direction of travel