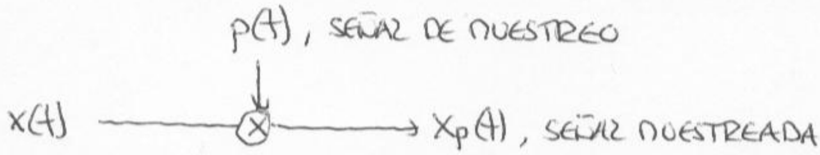
Tema 6: Muestreo y Reconstrucción

* Introducción

* Representación de señales de tiempo continuo a partir de sus muestras

① Muestreo ideal mediante un tren de impulsos

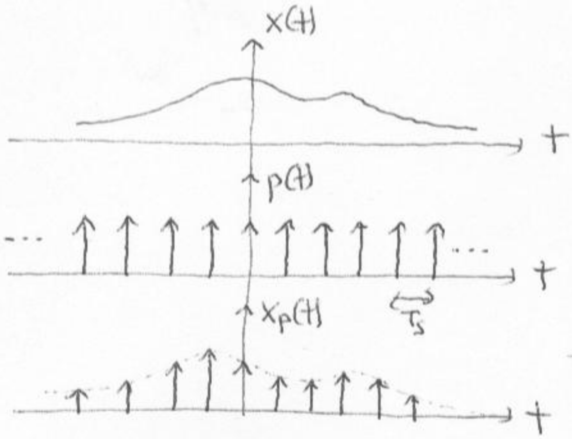
Modelo de muestreo:



Análisis en 't':

- $p(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT_s)$, periódica $T_s \Rightarrow \omega_s = 2\pi/T_s$

- $x_p(t) = x(t)\cdot p(t) = \sum_{k=-\infty}^{\infty} x(t)\,\delta(t-kT_s) = \sum_{k=-\infty}^{\infty} x(kT_s)\cdot\delta(t-kT_s)$



→ ¿Es posible recuperar $x(t)$ a partir de $x_p(t)$?

¿En qué condiciones?

Análisis en 'ω':

- $x(t) \xrightarrow{FT} X(j\omega)$

- $p(t) \xrightarrow{FT} P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} a_k\,\delta(\omega-k\omega_s) = \frac{2\pi}{T_s}\cdot\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$

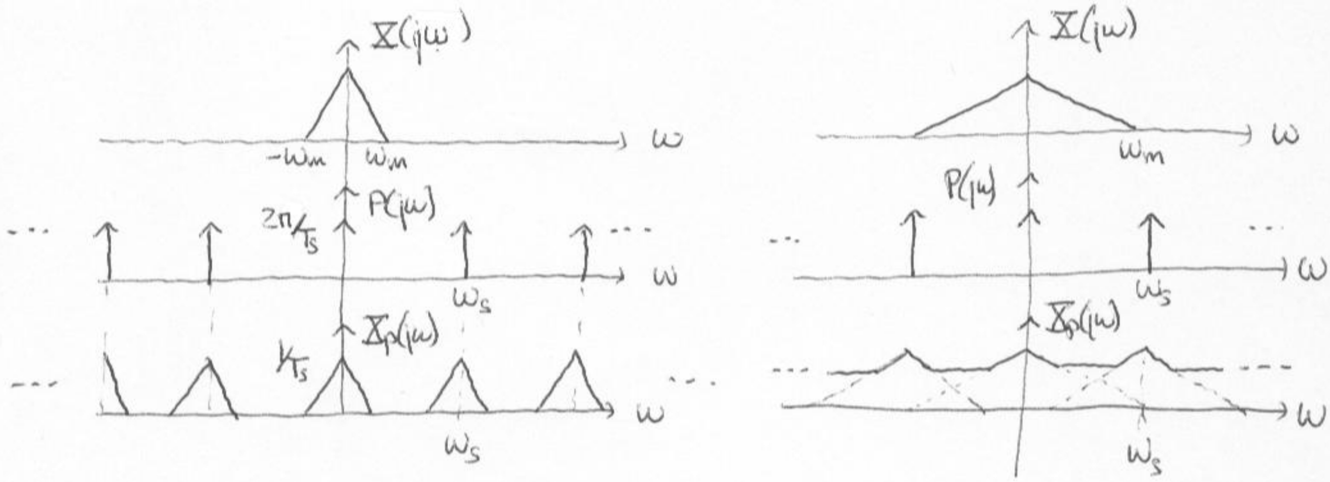
$$\xrightarrow{FS} a_k = \frac{1}{T_s}\int_{\langle T_s\rangle} p(t)\,e^{-jk\omega_s t}\,dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2}\delta(t)\,e^{-jk\omega_s t}\,dt = \frac{1}{T_s}\ ,\ \forall k$$

- $x_p(t) = x(t)\cdot p(t) \xrightarrow{FT} X_p(j\omega) = \frac{1}{2\pi}\cdot X(j\omega) * P(j\omega) =$

$$= \frac{1}{2\pi}\cdot\int_{-\infty}^{\infty} P(j\theta)\cdot X(j(\omega-\theta))d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j(\omega-\theta))\cdot\sum_{k=-\infty}^{\infty}\delta(\theta-k\omega_s)\cdot\frac{2\pi}{T_s}d\theta =$$

$$= \frac{1}{T_s}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))\cdot\delta(\theta-k\omega_s)\,d\theta = \frac{1}{T_s}\cdot\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))\cdot\int_{-\infty}^{\infty}\delta(\theta-k\omega_s)d\theta \Rightarrow$$

$$\Rightarrow X_p(j\omega) = \frac{1}{T_s}\cdot\sum_{k=-\infty}^{\infty} X(j(\omega-k\omega_s))$$



¿ES POSIBLE RECUPERAR $X(j\omega)$ A PARTIR DE $X_p(j\omega)$?

¿EN QUÉ CONDICIONES?

↓

TEOREMA DE MUESTREO O DE NYQUIST

PARA SEÑALES REALES → $x(t)$ LIMITADA EN BANDA POR $\omega_m \Rightarrow$ SI MUESTREO A $\underbrace{\omega_s}_{\text{pulsación de muestreo}} > \underbrace{2\omega_m}_{\text{pulsación de Nyquist}}$, PUEDO RECUPERAR $x(t)$ A PARTIR DE $x_p(t)$

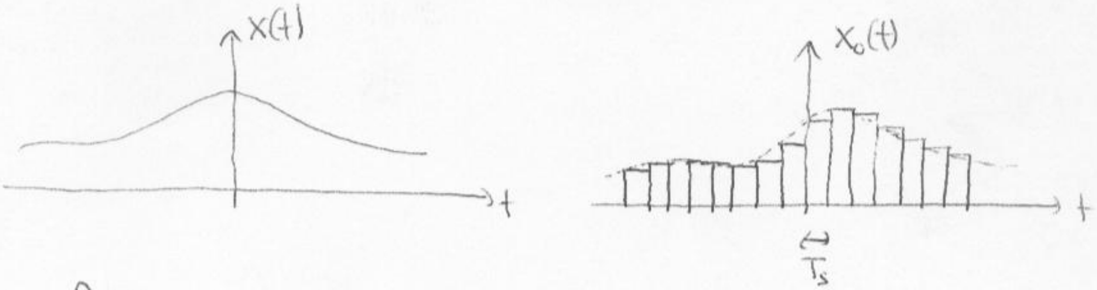
$X(j\omega)$ SE RECUPERA MEDIANTE FILTRADO PASO BAJO DE $X_p(j\omega)$. SI HAY SOLAPE DE ESPECTROS (ALIASING) NO SE PUEDE RECUPERAR

EN LA PRÁCTICA NO SUELEN UTILIZARSE FILTROS IDEALES: NO CAUSALES Y DE RESPUESTA AL IMPULSO OSCILATORIA.

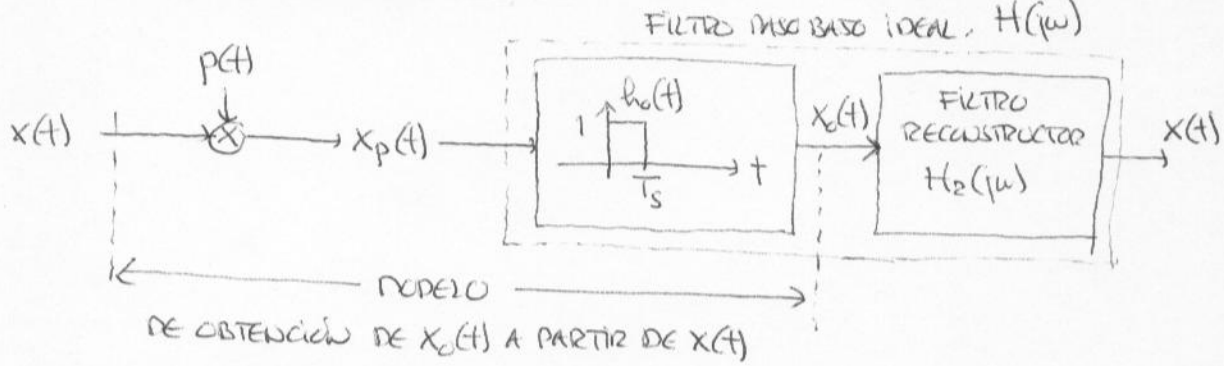
② Muestreo mediante un "zero order hold"

La generación y transmisión de impulsos no es viable en la práctica.

Aproximación alternativa:

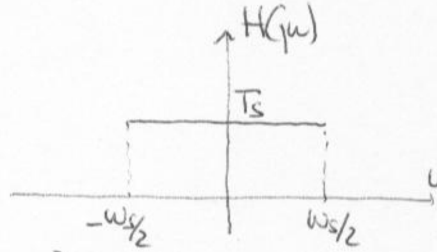


- Modelo de muestreo "zero order hold":



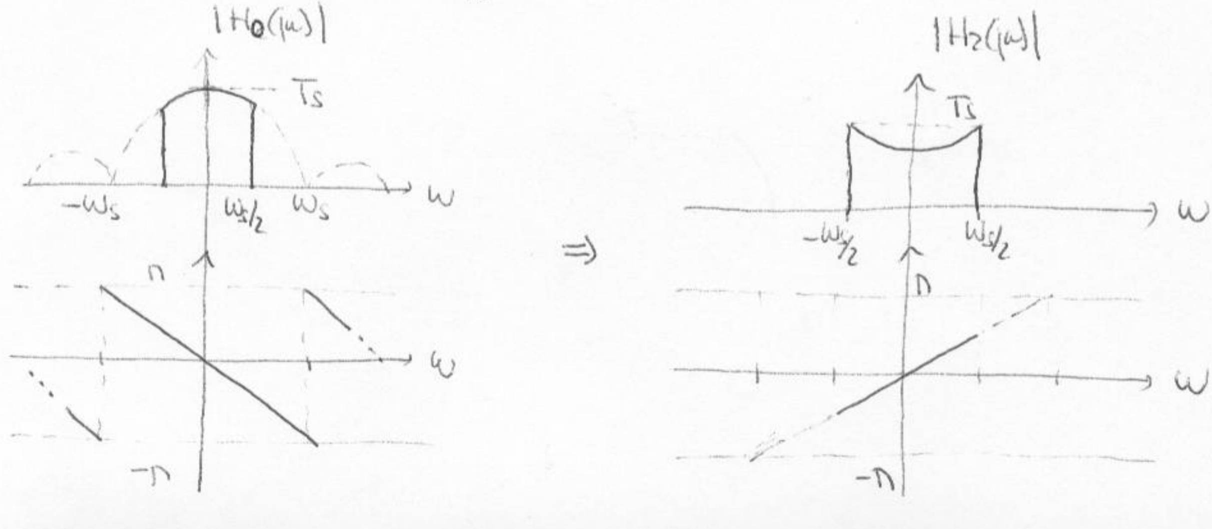
Efectivamente: $x_p(t) * h_0(t) = x_0(t)$

Para recuperar x(t) a partir de $x_0(t)$:



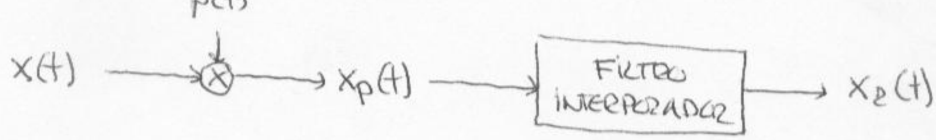
$$\left.\begin{array}{l} \cdot\ H(j\omega) = H_0(j\omega)\cdot \underbrace{H_2(j\omega)}_{?} \\ \cdot\ h_0(t) \xrightarrow{FT} H_0(j\omega) = e^{-j\omega\frac{T_s}{2}}\,\dfrac{2\,\mathrm{sen}(\omega T_s/2)}{\omega} \end{array}\right\} \Rightarrow$$

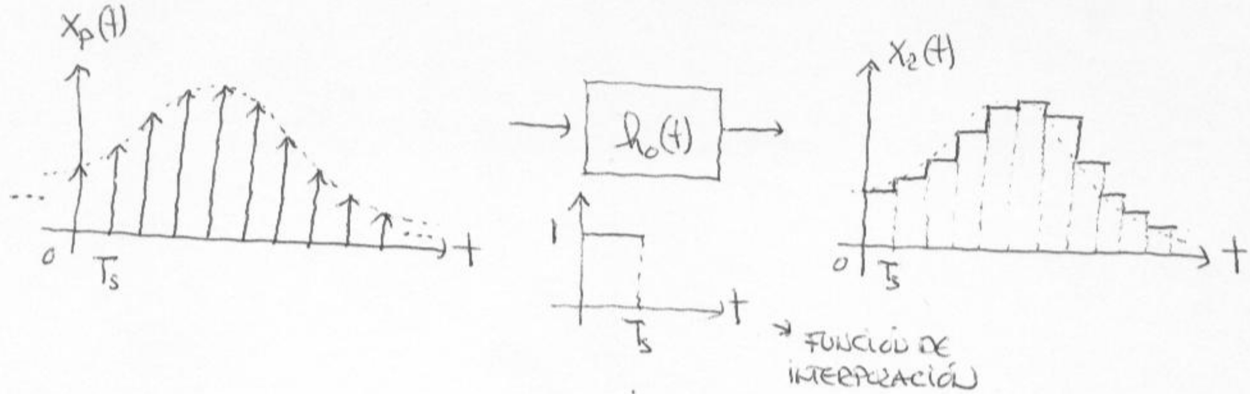
$$\Rightarrow H_2(j\omega) = \frac{e^{j\omega\frac{T_s}{2}}\cdot H(j\omega)}{\dfrac{2\,\mathrm{sen}(\omega T_s/2)}{\omega}}$$



* Reconstrucción por interpolación

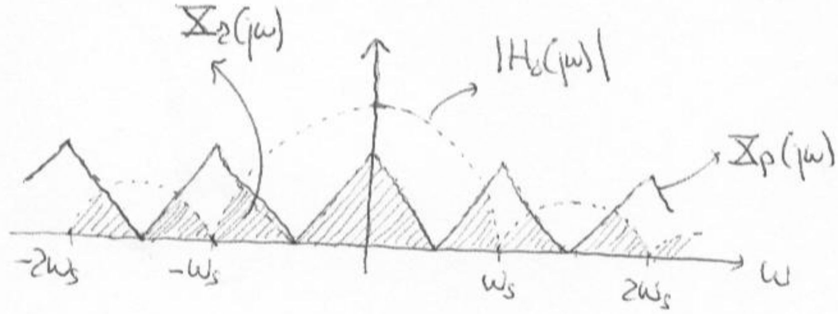
① Interpolación por replicación, o de orden cero



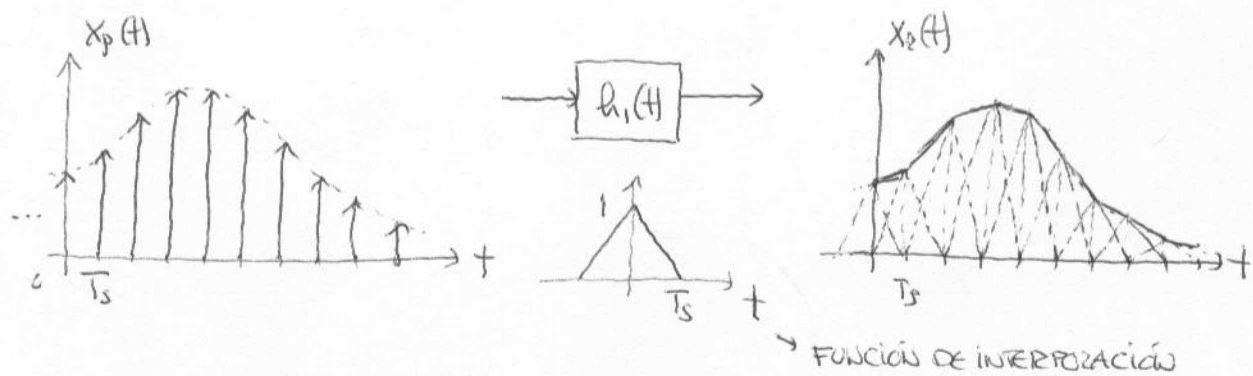


$\Big\downarrow$ FT

$$H_0(j\omega) = T_s \cdot \operatorname{sinc}\left(\frac{\omega}{\omega_s}\right) e^{-j\omega T_s/2}$$

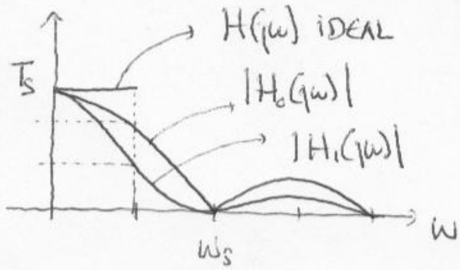


② Interpolación lineal, o de primer orden

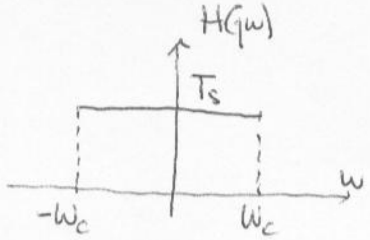


$$h_1(t) = h_0'(t) * h_0'(t) \cdot \frac{1}{T_s}, \quad \text{con} \quad h_0'(t) = u\left(t+\frac{T_s}{2}\right) - u\left(t-\frac{T_s}{2}\right) \Rightarrow$$

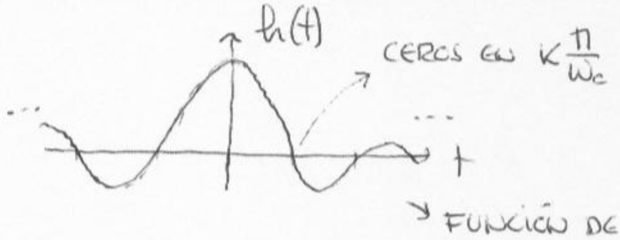
$$\Rightarrow H_1(j\omega) = \frac{1}{T_s} \cdot H_0'(j\omega) \cdot H_0'(j\omega) = \frac{1}{T_s}\left[T_s \operatorname{sinc}\left(\frac{\omega}{\omega_s}\right)\right]^2 = T_s \operatorname{sinc}^2\left(\frac{\omega}{\omega_s}\right)$$



③ INTERPOLACIÓN IDEAL



$$\xrightarrow{FT^{-1}} h(t) = T_s \cdot \frac{\operatorname{sen}(\omega_c t)}{\pi t} = \frac{T_s \omega_c}{\pi} \cdot \operatorname{sinc}\left(\frac{\omega_c t}{\pi}\right)$$

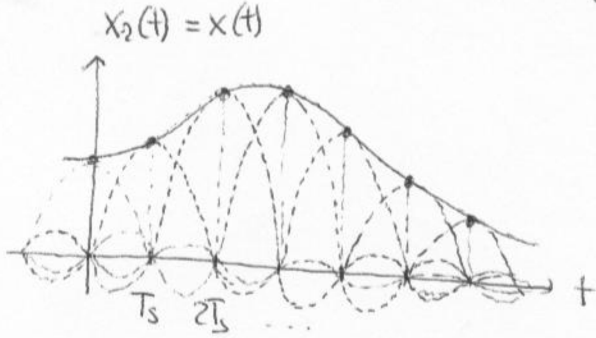


$\omega_m < \underline{\omega_c} < \omega_s - \omega_m$

$$x_2(t) = x(t) = x_p(t) * h(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \cdot \frac{T_s \omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c}{\pi}(t - kT_s)\right)$$

$$x_p(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \cdot \delta(t - kT_s)$$

· Si $\omega_c = \frac{\omega_s}{2} \Rightarrow$ $$x_2(t) = x(t) = \sum_{k=-\infty}^{\infty} x(kT_s) \cdot \underbrace{\operatorname{sinc}\left(\frac{1}{T_s}(t - kT_s)\right)}_{\text{CEROS EN } k'T_s \ (k' \neq k)}$$



• EL EFECTO DEL SUBMUESTREO : ALIASING DE FRECUENCIAS

- $\omega_s < 2\omega_m \Rightarrow$ ESPECTROS DE $X(j\omega)$ SOLAPADOS EN $X_p(j\omega) \rightarrow$ SUBMUESTREO

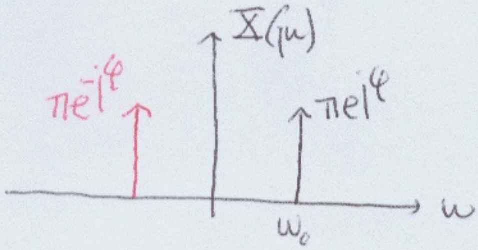
- EN ESTE CASO, $H(j\omega)$ IDEAL $\Rightarrow x_2(t) \neq x(t)$. SIN EMBARGO, SI $\omega_c = \frac{\omega_s}{2} \Rightarrow$

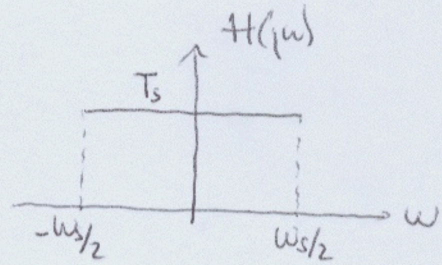
$$\Rightarrow x_2(k_0 T_s) = \sum_{k=-\infty}^{\infty} x(kT_s) \cdot \operatorname{sinc}(k_0 - k) = x(k_0 T_s), \ \forall k_0 \in \mathbb{Z}$$ $\Rightarrow$ ALGO DE $x(t)$ SÍ RECUPERO...

$$\operatorname{sinc}(k_0 - k) = \begin{cases} 1, & k = k_0 \\ 0, & k \neq k_0 \end{cases}$$

EJEMPLO:

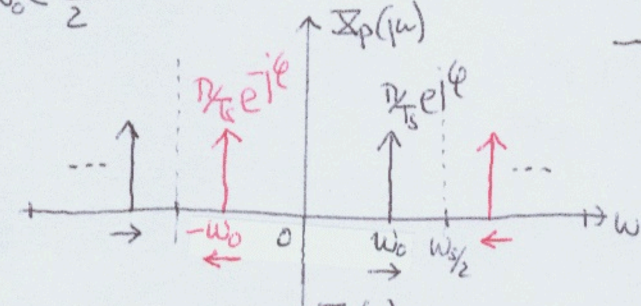
Sea $x(t) = \cos(\omega_0 t + \varphi)$, muestreada a ω_s y recuperada por filtrado ideal con $\omega_c = \omega_s/2$:



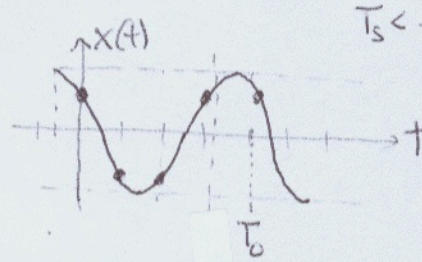


Si voy aumentando ω_0 manteniendo fija ω_s:

- $\omega_0 < \dfrac{\omega_s}{2}$

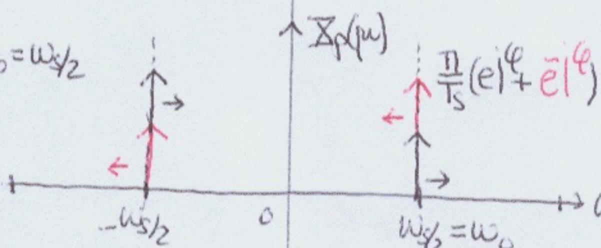


$\longrightarrow x_r(t) = x(t) = \cos(\omega_0 t + \varphi)$

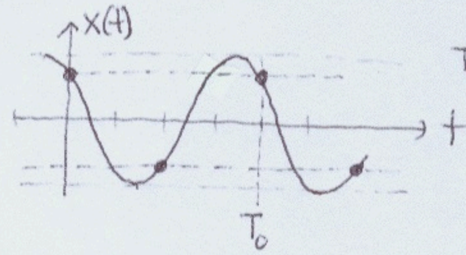


$T_s < \dfrac{T_0}{2}$

- $\omega_0 = \omega_s/2$

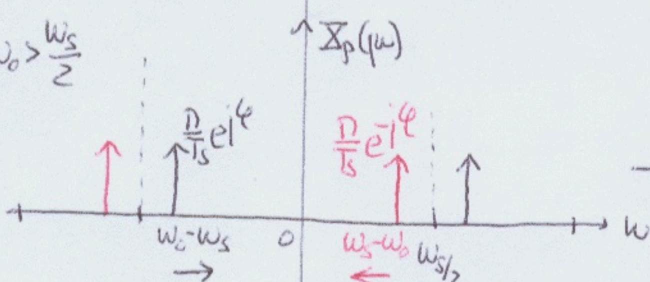


$\dfrac{\pi}{T_s}(e^{j\varphi} + e^{-j\varphi}) = \dfrac{\pi}{T_s} \cdot 2\mathrm{Re}[\varphi] \Rightarrow x_r(t) = 2\mathrm{Re}[\varphi]\cos(\omega_0 t) \neq x(t)$

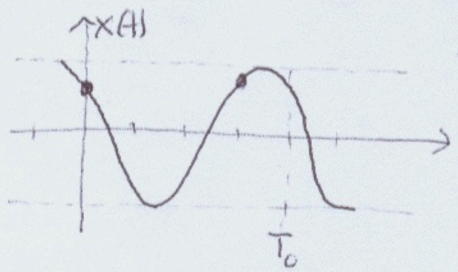


$T_s = \dfrac{T_0}{2}$

- $\omega_0 > \dfrac{\omega_s}{2}$

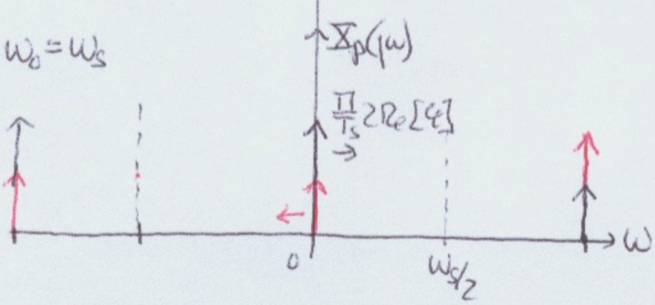


$\longrightarrow x_r(t) = \cos(\omega_0' t \underline{\underline{-}} \varphi) \neq x(t)$, $\omega_0' = \omega_s - \omega_0 < \omega_0$

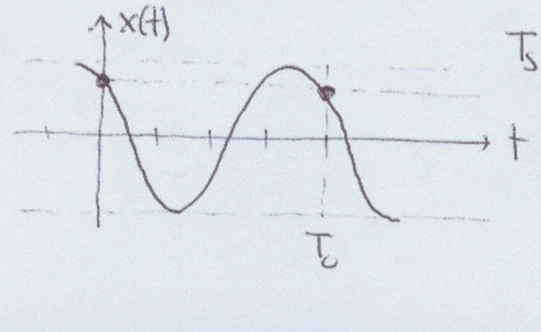


$T_s > T_0/2$

- $\omega_0 = \omega_s$



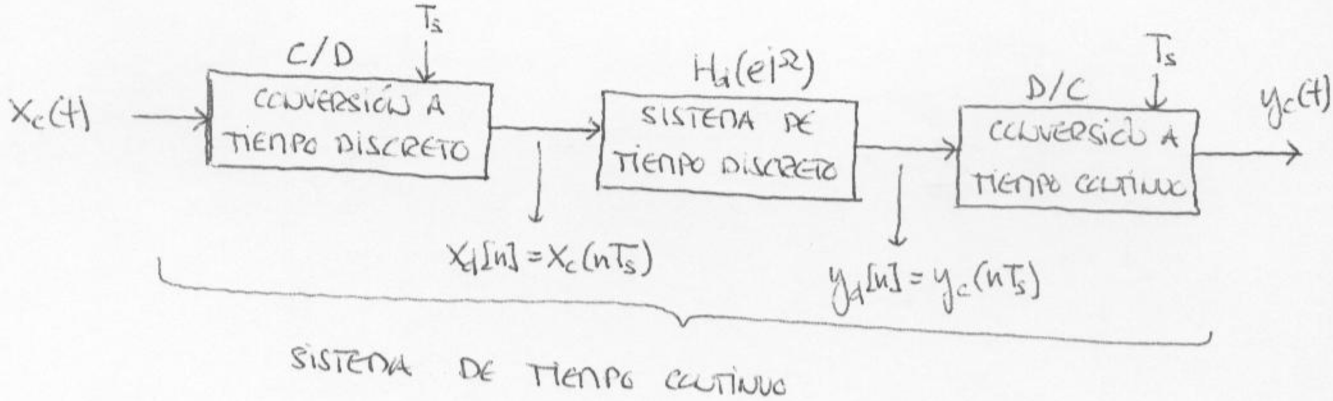
$\searrow x_r(t) = 2\mathrm{Re}[\varphi] \neq x(t)$



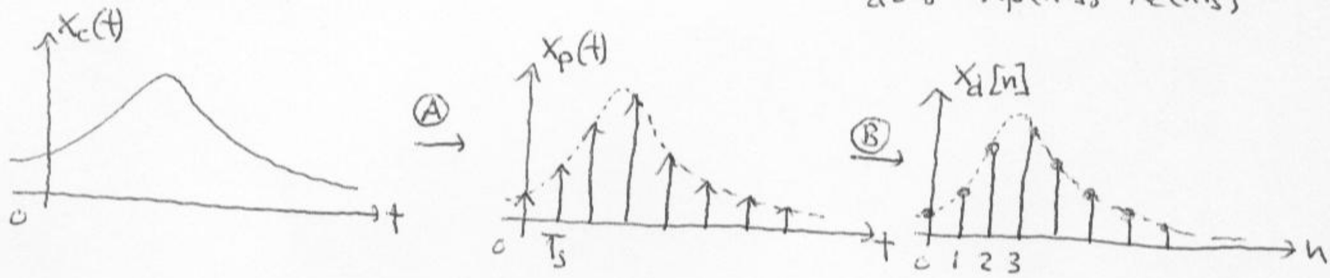
$T_s = T_0$

Efecto estroboscópico $\leftarrow$

* Tratamiento discreto de señales de tiempo continuo



$\boxed{C/D}$ → Muestreo periódico ideal (A) → $x_p(t)$ + Conversión tren de impulsos a secuencia (B) → $x_d[n] = x_p(nT_s) = x_c(nT_s)$



Análisis frecuencial:

(A) $\longrightarrow \; X_p(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$

(B) $\cdot \; x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT_s) \cdot \delta(t - nT_s) \xrightarrow{FT} X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT_s) \cdot e^{-j\omega n T_s}$

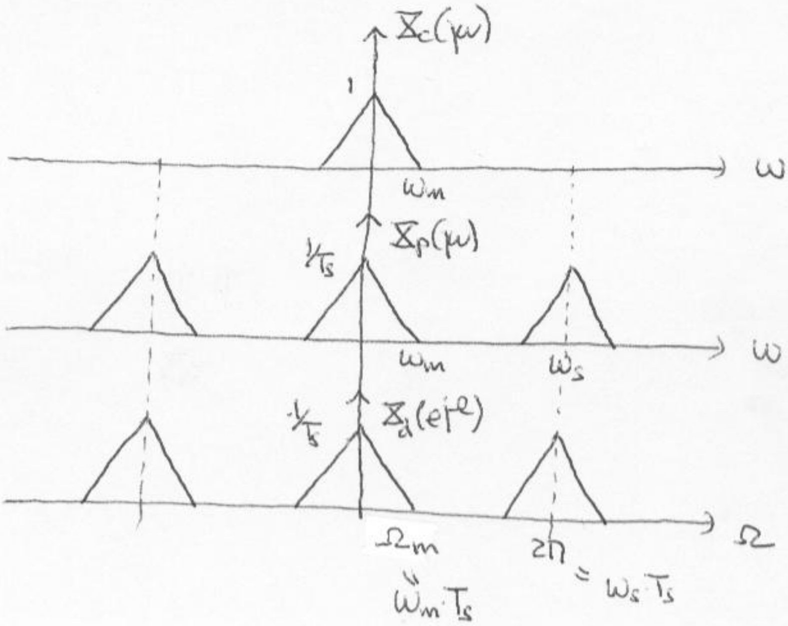
$\delta(t - t_0) \xrightarrow{FT} e^{-j\omega t_0}$

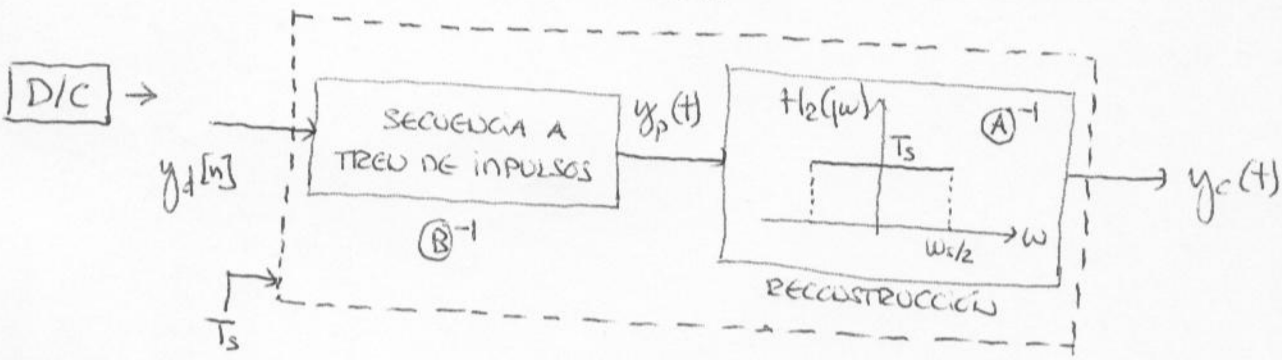
$\cdot \; x_d[n] = x_c(nT_s)$

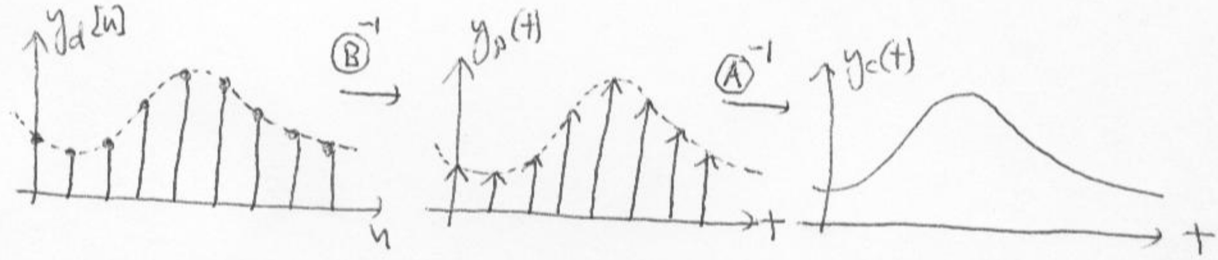
$\cdot \; x_d[n] \xrightarrow{DTFT} X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x_d[n] \cdot e^{-j\Omega n}$ $\Rightarrow$

$\Rightarrow X_d(e^{j\Omega}) = X_p(j\,\Omega/T_s) \Rightarrow$ $\quad \rightarrow \underbrace{X_d(e^{j2\pi}) = X_p(j\tfrac{2\pi}{T_s}) = X_p(j\omega_s)}_{\text{escalado por } T_s}$

$$\Rightarrow \boxed{X_d(e^{j\Omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_c\left(j(\Omega - 2k\pi)/T_s\right)}$$







ANÁLISIS FRECUENCIAL:

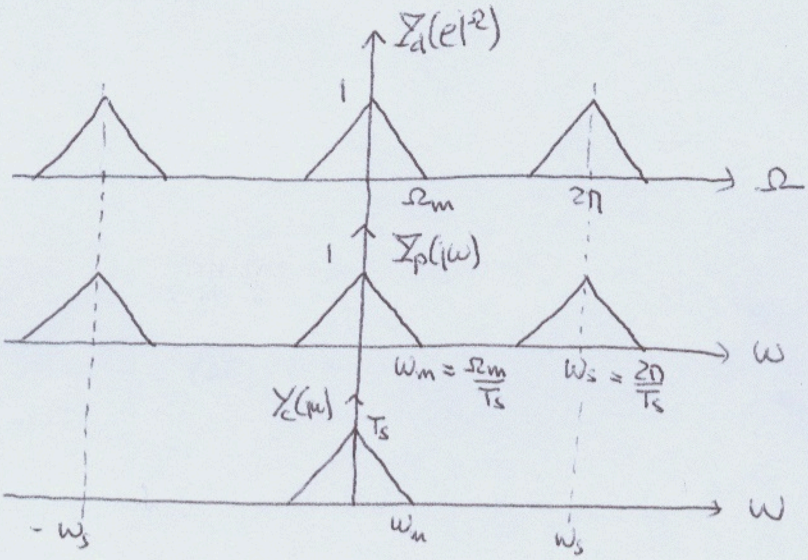
B^{-1}

$$\left.\begin{array}{l} \cdot\ y_d[n] \xrightarrow{\text{DTFT}} X_d(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y_d[n]\, e^{-j\Omega n} \\ \cdot\ y_p(t) = \sum_{n=-\infty}^{\infty} y_d[n]\cdot \delta(t-nT_s) \xrightarrow{\text{FT}} X_p(j\omega) = \sum_{n=-\infty}^{\infty} y_d[n]\, e^{-j\omega n T_s} \end{array}\right\} \Rightarrow$$

$$\Rightarrow X_p(j\omega) = X_d(e^{j\omega T_s})$$

A^{-1}

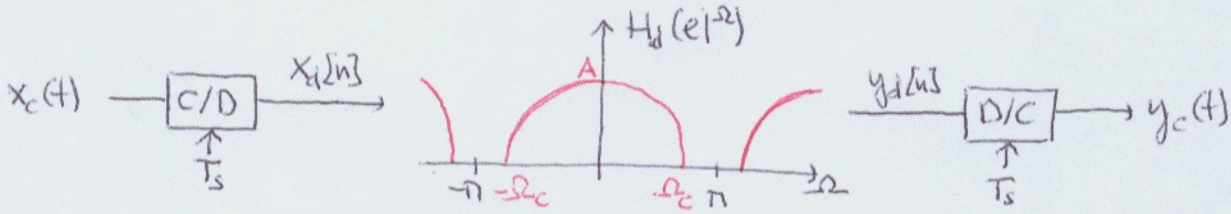
$$\cdot\ y_c(t) = y_p(t) * h_2(t) = \sum_{n=-\infty}^{\infty} y_d[n]\cdot \operatorname{sinc}\left(\frac{1}{T_s}(t-nT_s)\right)$$

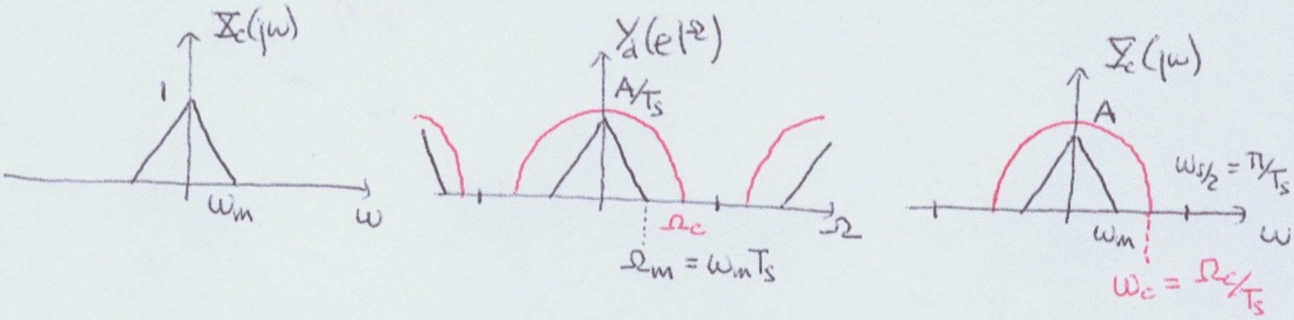


Efecto de recuperar (D/C) con $T_s' \neq T_s \rightarrow Y_c(j\omega) \,/ \begin{cases} 1 \rightarrow T_s'/T_s \\ \omega_m \rightarrow \omega_m \cdot T_s/T_s' \end{cases} \Rightarrow$

$\Rightarrow$ escalado en 't' de $y_c(t)$ respecto de $x_c(t)$

- Realización discreta de sistemas LTI de tiempo continuo





$$Y_c(j\omega) = X_c(j\omega)\cdot H_c(j\omega) \Rightarrow H_c(j\omega) = \begin{cases} H_d(e^{j\omega T_s}), & |\omega| < \omega_s/2 \\ 0, & \text{resto} \end{cases}, \Leftrightarrow \omega_s > 2\omega_m$$

- Obtención de $h_d[n]$: (2 métodos)

Dada una relación $y_c(t) = f\{x_c(t)\}$, obtengo $H_c(j\omega)$

1. Obtengo $H_d(e^{j\Omega}) = H_c(j\Omega/T_s)$, $|\Omega| < \pi$, donde $T_s \,/\, \omega_s > 2\omega_m$

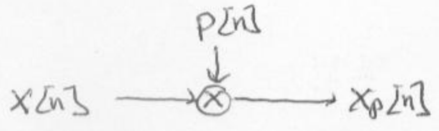
   Obtengo $h_d[n] \xleftarrow{DTFT^{-1}} H_d(e^{j\Omega})$

2. Obtengo la respuesta, $y_{c1}(t)$ a la entrada $x_1(t) = \dfrac{sen\left(\frac{\pi t}{T_s}\right)}{\pi t} = \dfrac{1}{T_s}\, sinc\left(\dfrac{t}{T_s}\right)$

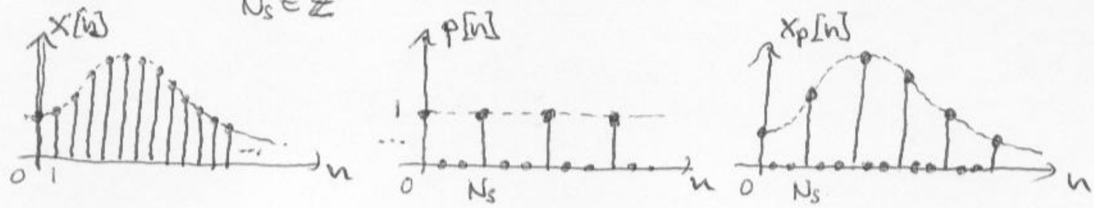
   Dado que $x_d[n] = x_{c1}(nT_s) = \dfrac{1}{T_s}\delta[n] \Rightarrow y_{c1}(nT_s) = y_d[n] = \dfrac{1}{T_s}\cdot h_d[n]$

* Muestreo de señales discretas

- Muestreo ideal mediante un tren de impulsos



$$p[n] = \sum_{k=-\infty}^{\infty} \delta[n - \underbrace{kN_s}_{N_s \in \mathbb{Z}^+}] \Rightarrow x_p[n] = x[n] \cdot p[n] = \sum_{k=-\infty}^{\infty} x[kN_s] \cdot \delta[n - kN_s]$$



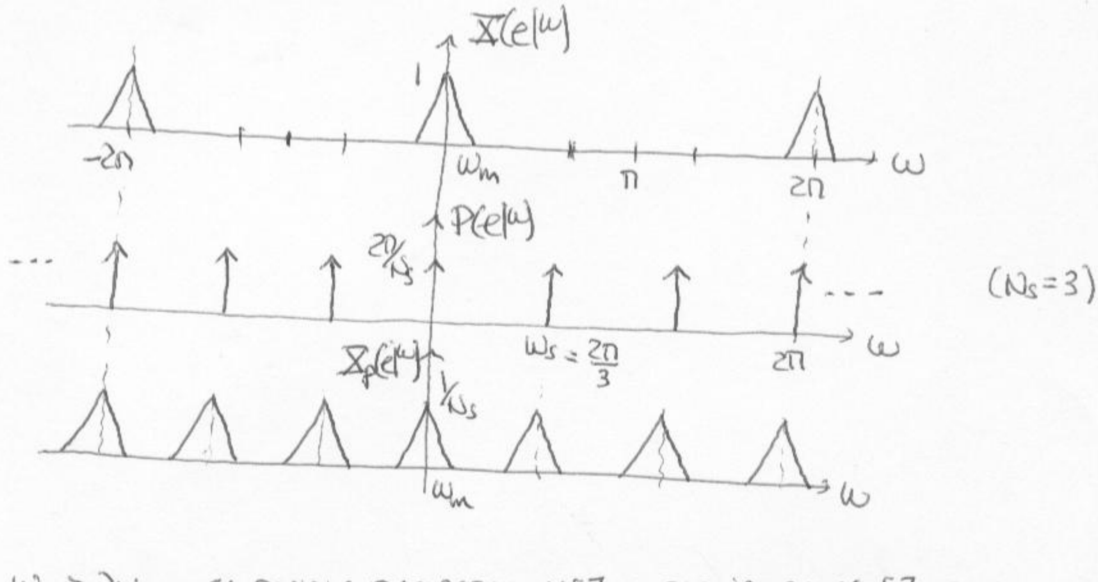
Análisis frecuencial:

$$\cdot\ x[n] \xrightarrow{DTFT} X(e^{j\omega})$$

$$\cdot\ p[n] \xrightarrow{DTFT} P(e^{j\omega}) = \frac{2\pi}{N_s} \cdot \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \Bigg\} \Rightarrow$$

$$\Rightarrow x_p[n] \xrightarrow{DTFT} X_p(e^{j\omega}) = \frac{1}{2\pi} \cdot X(e^{j\omega}) \circledast P(e^{j\omega}) \Rightarrow$$

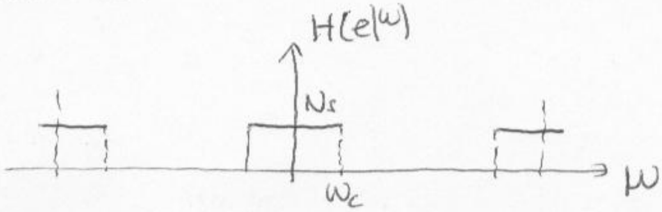
$$\Rightarrow X_p(e^{j\omega}) = \frac{1}{N_s} \cdot \sum_{k=0}^{N_s-1} X(e^{j(\omega - k\omega_s)})$$



· Si $\omega_s > 2\omega_m$, es posible recuperar $x[n]$ a partir de $x_p[n]$ mediante filtrado paso bajo → problema 7.15

· Restricción: $\frac{2\pi}{N_s} > \omega_m$, pero $N_s \in \mathbb{Z}^+$

- RECONSTRUCCIÓN:



$$\omega_m < \omega_c < \omega_s - \omega_m$$

$$H(e^{j\omega}) \xrightarrow{DTFT^{-1}} h[n] = N_s \cdot \frac{\operatorname{sen} \omega_c n}{\pi n} = \frac{N_s \omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

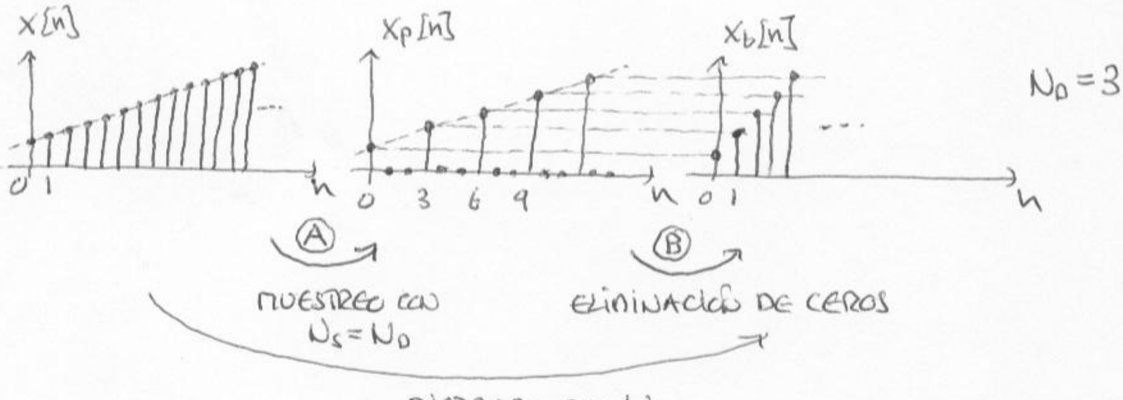
$$x_2[n] = x_p[n] * h[n] = \sum_{k=-\infty}^{\infty} x[kN_s] \cdot h[n-kN_s] =$$

$$= \sum_{k=-\infty}^{\infty} x[kN_s] \cdot \frac{N_s \omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c (n-kN_s)}{\pi}\right)$$

- Si $\omega_c = \frac{\omega_s}{2} \Rightarrow h[n] = \operatorname{sinc}\left(\frac{n}{N_s}\right) \Rightarrow x_2[n] = \sum_{k=-\infty}^{\infty} x[kN_s] \cdot \operatorname{sinc}\left(\frac{n-kN_s}{N_s}\right) \Rightarrow$

$\Rightarrow x_2[kN_s] = x[kN_s]$, $\forall k \in \mathbb{Z}$, CON INDEPENDENCIA DE ω_s

- DIEZMADO (DOWNSAMPLING)



- $x_b[n] = x_p[nN_D] = x[nN_D] \longrightarrow$ ESCALADO POR N_D
- ANÁLISIS FRECUENCIAL:

Ⓐ $\rightarrow X_p(e^{j\omega}) = \frac{1}{N_D} \cdot \sum_{k=0}^{N_D-1} X(e^{j(\omega - k\frac{2\pi}{N_D})})$

Ⓑ · $x_p[n] \xrightarrow{DTFT} X_p(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_p[n] e^{-j\omega n}$

· $x_b[n] \xrightarrow{DTFT} X_b(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_b[n] e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_p[nN_D] e^{-j\omega n} \underset{k = nN_D}{=}$

$= \sum_{k=-\infty}^{\infty} x_p[k] \cdot e^{-j\omega \frac{k}{N_D}} = \sum_{k=-\infty}^{\infty} x_p[k] e^{-j\omega \frac{k}{N_D}}$ $\Rightarrow$

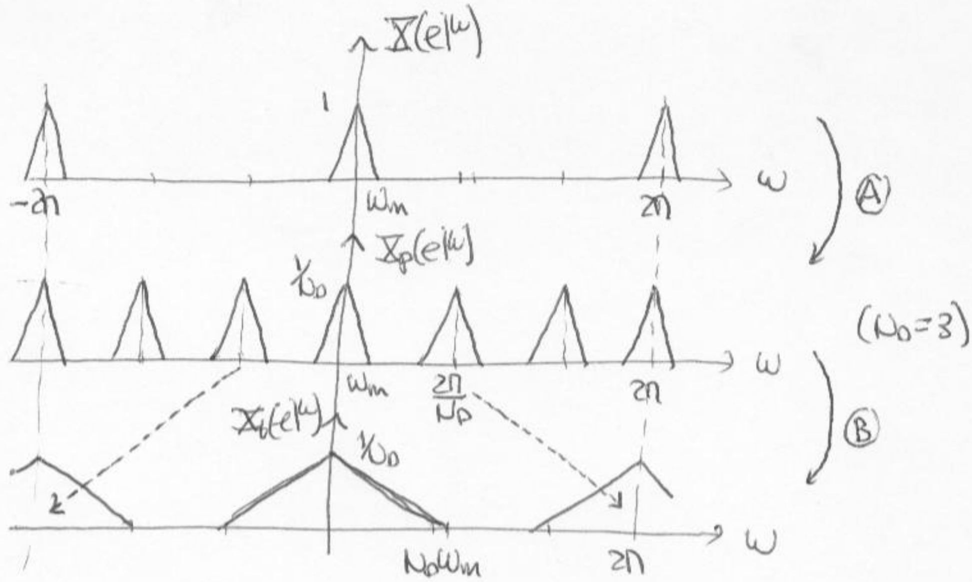
k MÚLTIPLO DE N_D

$x_p[k] = 0$ SI k NO ES MÚLTIPLO DE N_D

$$\Rightarrow \underbrace{X_b(e^{j\omega}) = X_p(e^{j\omega/N_D})}_{\text{ESCALADO EN } \omega} = \frac{1}{N_D} \cdot \sum_{k=0}^{N_D-1} X\left(e^{j\frac{\omega-2k\pi}{N_D}}\right)$$

↓

¡OJO! : SÓLO SI LO QUE ELIMINO SON CEROS



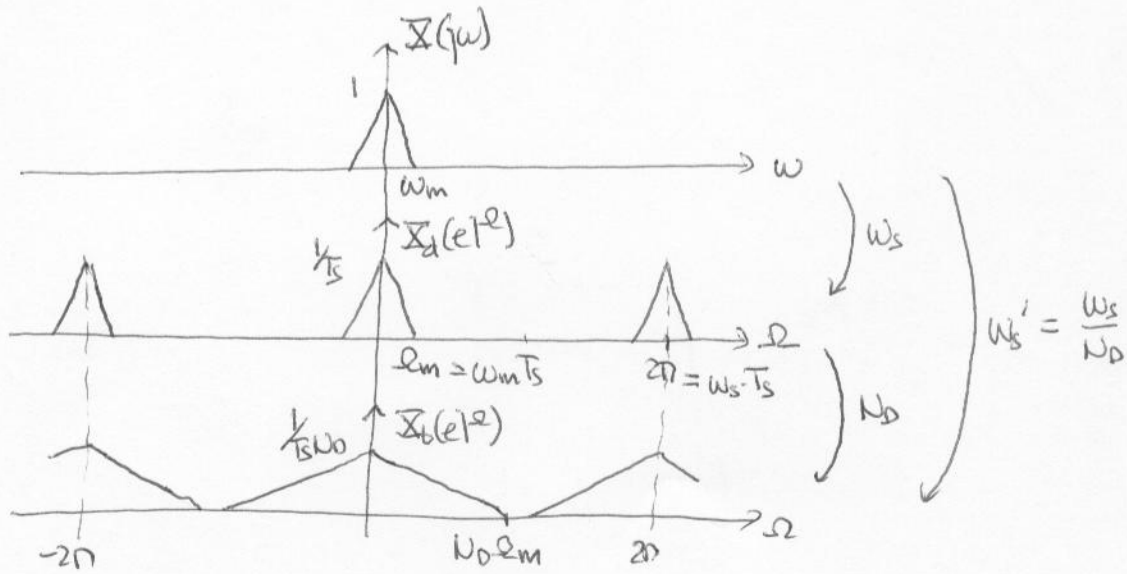
SI NO HAY SOLAPE ESPECTRAL , $X_b(e^{j\omega}) = \frac{1}{N_D} \cdot X(e^{j\frac{\omega}{N_D}})$

DOWNSAMPLING:

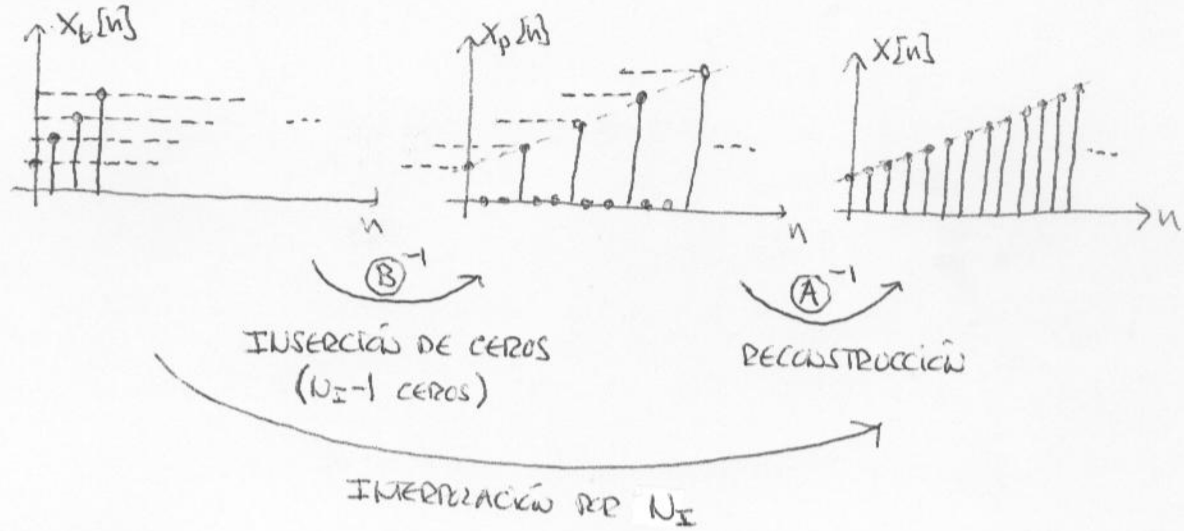
$x(t) \xrightarrow[T_s]{} x_d[n] \xrightarrow[N_D]{} x_b[n]$: si

N_D → SI NO SE PRODUCE SOLAPE ESPECTRAL ⇒

⇒ $x(t)$ ESTABA SOBREMUESTREADA ⇒ PUEDO REDUCIR MUESTRAS SIN PERDER INFORMACIÓN, O HABER MUESTREADO CON $T_s' = T_s \cdot N_D$



- INTERPOLACIÓN (UPSAMPLING)

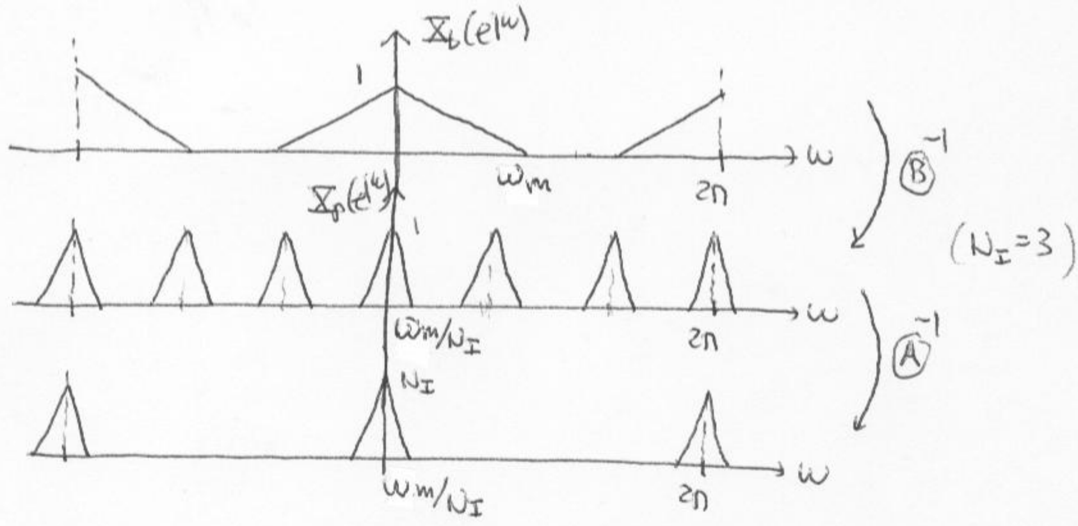


ANÁLISIS FRECUENCIAL:

$$x_p[n] = \begin{cases} x_b[n/N_I], & n \text{ múltiplo de } N_I \\ 0 & \text{resto} \end{cases} \Rightarrow$$

$$\Rightarrow \quad X_p(e^{j\omega}) = X_b(e^{jN_I\omega}) \quad \rightarrow \text{PROPIEDAD DE ESCALADO DTFT}$$

$$X(e^{j\omega}) = X_p(e^{j\omega}) \cdot \underbrace{H(e^{j\omega})}_{\text{FILTRO PASO BAJO RECONSTRUCTOR}}$$



- MUESTREO POR FACTORES NO ENTEROS
  (INTERPOLACIÓN Y DIEZMADO)

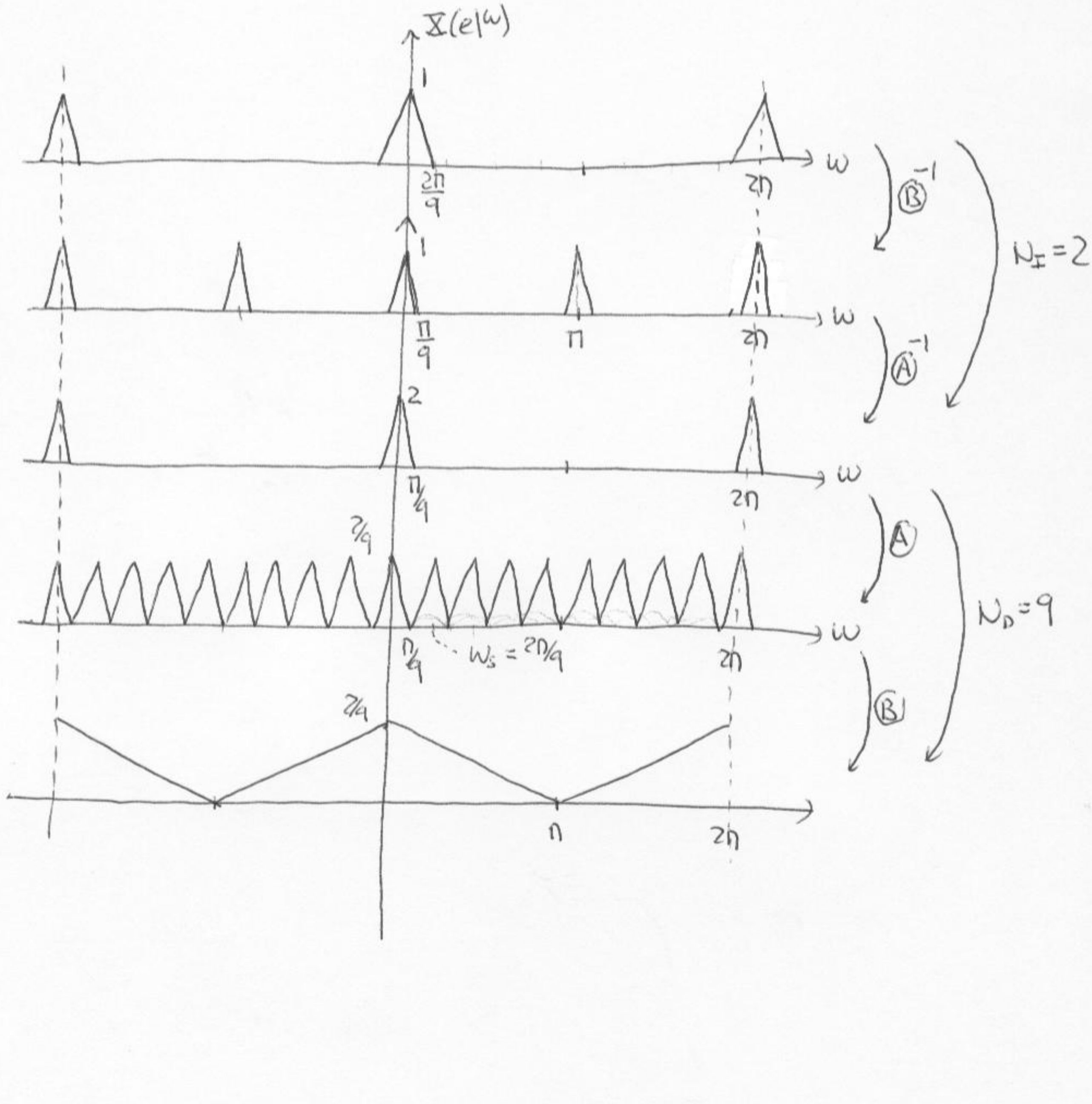
EJEMPLO:

Sea $x[n]$ / $X(e^{j\omega}) = 0$, $\frac{2\pi}{9} < |\omega| < \pi \Rightarrow \omega_m < \frac{2\pi}{9}$

$$\frac{2\pi}{N_s} > 2\omega_m \Rightarrow \frac{2\pi}{N_s} \geq \frac{2\pi}{9} \cdot 2 \Rightarrow N_s \leq \frac{9}{2}$$

- Restricción $N_s \in \mathbb{Z}^+ \Rightarrow N_s \leq 4$
- Alternativa:

$N_s \leq \frac{9}{2} \Rightarrow$ interpolar con $N_I = 2$ y diezmar con $N_D = 9$:



PROBLEMA 7.3

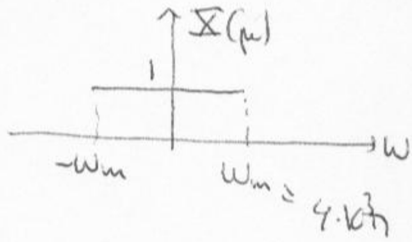
OBTENER LA PULSACIÓN DE NYQUIST DE LAS SEÑALES

a) $x(t) = 1 + \cos(2 \cdot 10^3 \pi t) + \text{sen}(4 \cdot 10^3 \pi t)$

$$x(t) \xrightarrow{FT} X(j\omega) = 2\pi\delta(\omega) + \pi\left[\delta(\omega - 2\cdot10^3\pi) + \delta(\omega + 2\cdot10^3\pi)\right] + \frac{\pi}{j}\left[\delta(\omega - 4\cdot10^3\pi) + \delta(\omega + 4\cdot10^3\pi)\right] \Rightarrow$$

$$\Rightarrow \omega_m = 4\cdot10^3\pi \Rightarrow \omega_s > 2\omega_m = \underline{8\cdot10^3\pi \text{ rad/s}}$$

b) $x(t) = \dfrac{\text{sen}(4\cdot10^3\pi t)}{\pi t} \xrightarrow{FT}$

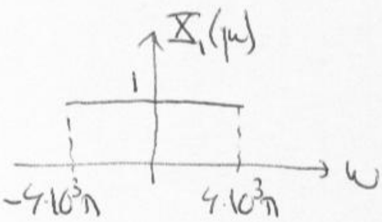


$\Rightarrow$

$$\Rightarrow \omega_s > 2\omega_m = 8\cdot10^3\pi \text{ rad/s}$$

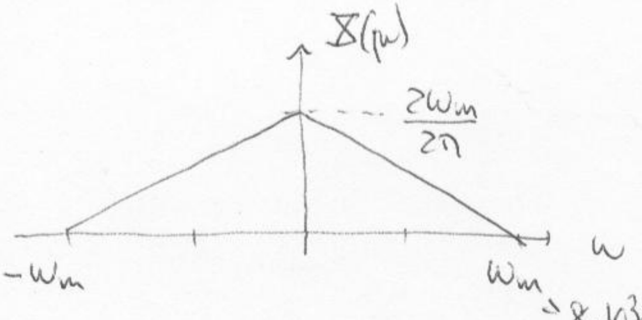
c) $x(t) = \left[\dfrac{\text{sen}(4\cdot10^3\pi t)}{\pi t}\right]^2$

SABIENDO QUE $x_1(t) = \dfrac{\text{sen}(4\cdot10^3\pi t)}{\pi t} \xrightarrow{FT}$



, POR LA PROPIEDAD DE MULTIPLICACIÓN:

$$x(t) = x_1(t)\cdot x_1(t) \xrightarrow{FT} X(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_1(j\omega) ;$$



$= 8\cdot10^3\pi \Rightarrow \omega_s > 2\omega_m = 16\cdot10^3\pi$

PROBLEMA 7.4

$\Rightarrow \omega_m < \frac{\omega_0}{2}$

$x(t) \,/\, \omega_s\big|_{min} = \omega_0$. DETERMINAR $\omega_s\big|_{min}$ PARA:

a) $x(t) + x(t-1) \xrightarrow{FT} X(j\omega) - e^{-j\omega} \cdot X(j\omega) =$

$= (1 - e^{-j\omega}) \cdot X(j\omega) \Rightarrow \omega_m$ NO VARÍA $\Rightarrow \omega_s\big|_{min} = \underline{\omega_0}$

b) $\frac{dx(t)}{dt} \xrightarrow{FT} j\omega X(j\omega) \Rightarrow \omega_m$ NO VARÍA $\Rightarrow \omega_s\big|_{min} = \underline{\omega_0}$

c) $x(t) \cdot x(t) \xrightarrow{FT} \frac{1}{2\pi} X(j\omega) * X(j\omega) \Rightarrow$

$\Rightarrow$ si $X(j\omega) \neq 0$ en $[-\omega_1, \omega_2]$, $X(j\omega) * X(j\omega) \neq 0$ en $[-2\omega_1, 2\omega_2] \Rightarrow$

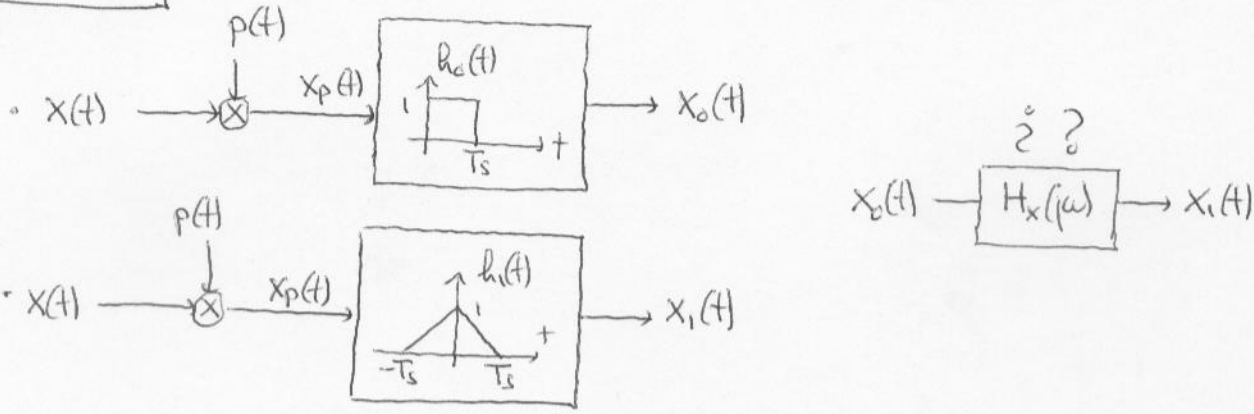
$\Rightarrow \omega_m$ SE DUPLICA $\Rightarrow \omega_s\big|_{min} = \underline{2\omega_0}$

d) $x(t) \cdot \cos\omega_0 t \xrightarrow{FT} \frac{1}{2\pi} \cdot X(j\omega) * [\pi\,\delta(\omega-\omega_0) + \pi\,\delta(\omega+\omega_0)] =$

$= \frac{1}{2\pi} \{ X(j(\omega-\omega_0)) + X(j(\omega-\omega_0)) \} \cdot \pi \Rightarrow$

$\Rightarrow \omega_m'$ PASA A SER $\omega_0 + \omega_m \leq \omega_0 + \frac{\omega_0}{2} \leq \frac{3}{2}\omega_0 \Rightarrow \omega_s\big|_{min} = \underline{3\omega_0}$

PROBLEMA 7.7



$$X_0(j\omega)\cdot H_x(j\omega) = X_1(j\omega) \Rightarrow H_x(j\omega) = \frac{X_1(j\omega)}{X_0(j\omega)}$$

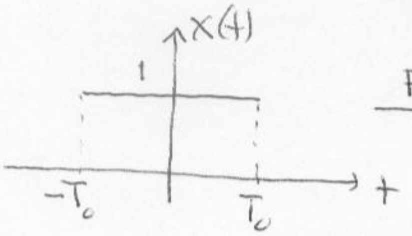
$$\left.\begin{array}{l} \rightarrow X_0(j\omega) = X_p(j\omega)\cdot H_0(j\omega) \\ \rightarrow X_1(j\omega) = X_p(j\omega)\cdot H_1(j\omega) \end{array}\right\} \Rightarrow H_x(j\omega) = \frac{H_1(j\omega)}{H_0(j\omega)}$$

$$\left.\begin{array}{l} h_0(t) \xrightarrow{\quad FT \quad} H_0(j\omega) = e^{-j\omega\frac{T_s}{2}}\cdot T_s\ \mathrm{sinc}\left(\frac{\omega}{\omega_s}\right) \\ h_1(t) \xrightarrow{\quad FT \quad} H_1(j\omega) = T_s\cdot \mathrm{sinc}^2\left(\frac{\omega}{\omega_s}\right) \end{array}\right\} \Rightarrow$$

$$\Rightarrow \boxed{H_x(j\omega) = e^{j\omega\frac{T_s}{2}}\cdot \mathrm{sinc}\left(\frac{\omega}{\omega_s}\right)}$$

PROBLEMA 7.10

a) $x(t) = u(t+T_0) - u(t-T_0)$ PUEDE MUESTREARSE SIN ALIASING CON $T_s < 2T_0$.

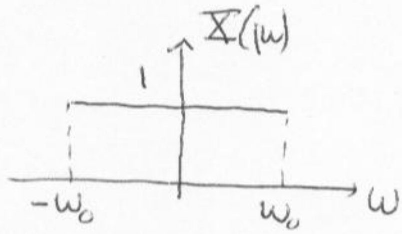


$\xrightarrow{FT} X(j\omega) \,/\, \nexists\, \omega_m \,,\; X(j\omega) = 0 \quad |\omega| > \omega_m \Rightarrow$

$\Rightarrow$ NO LIMITADA EN BANDA $\Rightarrow$ FALSO

b) $x(t) \,/\, X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ PUEDE MUESTREARSE SIN ALIASING CON

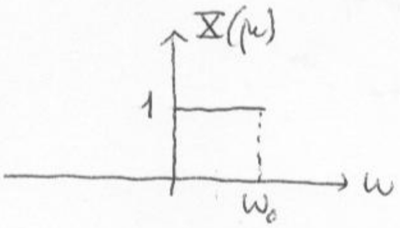
$T_s < \pi/\omega_0$



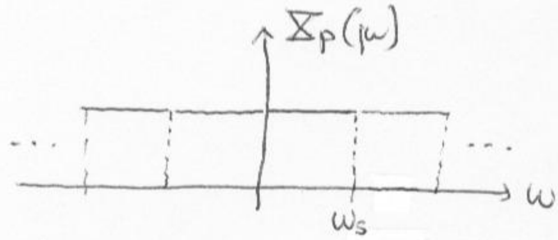
$\Rightarrow \omega_m = \omega_0 \Rightarrow \omega_s > 2\omega_m \Rightarrow \omega_s > 2\omega_0 \Rightarrow$

$\Rightarrow \dfrac{2\pi}{T_s} > 2\omega_0 \Rightarrow T_s < \dfrac{\pi}{\omega_0}$, VERDADERO

c) $x(t) \,/\, X(j\omega) = u(\omega) - u(\omega - \omega_0)$ PUEDE MUESTREARSE SIN ALIASING CON $T_s < \dfrac{2\pi}{\omega_0}$



$\longrightarrow$



NO HAY SOLAPE SI $\omega_s > \omega_0 \Rightarrow$

$\Rightarrow \dfrac{2\pi}{T_s} > \omega_0 \Rightarrow T_s < \dfrac{2\pi}{\omega_0} \Rightarrow$ VERDADERO

TEOREMA DE NYQUIST VS SEÑALES NO REALES

PROBLEMA 7.13

OBTENER $h_d[n]$ QUE IMPLEMENTA EL SISTEMA $y_c(t) = x_c(t-2T_s)$ PARA SEÑALES $x_c(t)$ / $W_m < 2W_s$.

* MÉTODO ①:

· $y_c(t) = x_c(t-2T_s) \Rightarrow Y_c(j\omega) = e^{-2j\omega T_s} X_c(j\omega) \Rightarrow H_c(j\omega) = e^{-2j\omega T_s}$

· $H_d(e^{j\Omega}) = H_c(j\Omega/T_s) = e^{-2j\Omega}, \quad |\Omega| < \pi$

$$\delta[n-n_0] \xrightarrow{DTFT} e^{-j\omega n_0} \Rightarrow H_d(e^{j\Omega}) = e^{-2j\Omega} \xrightarrow{DTFT^{-1}} h_d[n] = \delta[n-2]$$

* MÉTODO ②:

$$x_c(t) = \frac{1}{T_s}\,\text{sinc}\left(\frac{t}{T_s}\right) \Rightarrow y_c(t) = \frac{1}{T_s}\,\text{sinc}\left(\frac{t-2T_s}{T_s}\right) \Rightarrow x_d[n] = \frac{1}{T_s}\,\delta[n] \Rightarrow$$

$$\Rightarrow h_d[n] = T_s \cdot y_c(nT_s) = \text{sinc}\left(\frac{nT_s-2T_s}{T_s}\right) = \text{sinc}(n-2) = \delta[n-2]$$

PROBLEMA 7.14

OBTENER $h_d[n]$ QUE IMPLEMENTA EL SISTEMA $y_c(t) = \frac{d}{dt} x_c\left(t - \frac{T_s}{2}\right)$ PARA SEÑALES $x_c(t) / \omega_m < 2\omega_s$.

* MÉTODO 1 :

$$\cdot\ Y_c(j\omega) = j\omega \cdot e^{-j\omega \frac{T_s}{2}} \cdot X_c(j\omega) \Rightarrow H_c(j\omega) = j\omega e^{-j\omega \frac{T_s}{2}}$$

$$\cdot\ H_d(e^{j\Omega}) = H_c(j\Omega/T_s) = j\frac{\Omega}{T_s} e^{-j\frac{\Omega}{2}} = \frac{\Omega}{T_s} e^{-j\left(\frac{\Omega}{2} - \frac{\pi}{2}\right)}$$

$$h_d[n] = \frac{1}{2\pi} \int_{\langle 2\pi \rangle} \frac{\Omega}{T_s} e^{-j\left(\frac{\Omega}{2} - \frac{\pi}{2}\right)} e^{j\Omega n} d\Omega = \cdots$$

* MÉTODO 2 :

$$\underbrace{\cdot\ x_c(t) = \frac{\operatorname{sen}(\pi t/T_s)}{\pi t}}_{\Downarrow} \Rightarrow x_d[n] = x_c(nT_s) = \frac{1}{T_s} \delta[n]$$

$$y_c(t) = \underbrace{\frac{\cos\left(\pi(t - T_s/2)/T_s\right)}{t - T_s/2}}_{A} - \underbrace{\frac{\operatorname{sen}\left(\pi(t - T_s/2)/T_s\right)}{\pi(t - T_s/2)}}_{B} \Rightarrow$$

$$\Rightarrow y_d[n] = y_c(nT_s) = -\frac{\operatorname{sen}\left(\pi\left(n - \frac{1}{2}\right)\right)}{\pi T_s^2 \left(n - \frac{1}{2}\right)^2}$$

$$\cdot\ h_d[n] = T_s \cdot y_d[n] = \frac{-\operatorname{sen}\left(\pi\left(n - \frac{1}{2}\right)\right)}{\pi T_s \left(n - \frac{1}{2}\right)^2} = \boxed{\frac{(-1)^n}{\pi T_s \left(n - \frac{1}{2}\right)^2}}$$

PROBLEMA 7.15

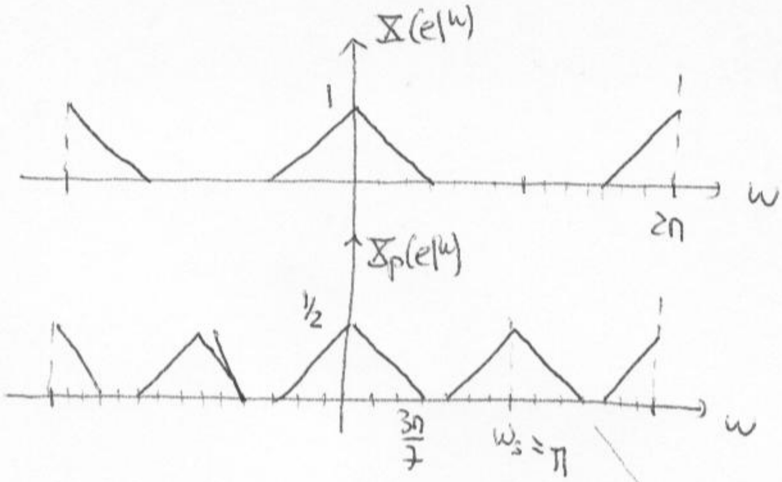
$$x[n] \longrightarrow g[n] = \sum_{k=-\infty}^{\infty} x[n]\cdot\delta[n-kN_s] \quad , \quad X(e^{jw}) = 0 \ , \ \frac{3\pi}{7} \le |w| \le \pi$$

¿ N_s MAX / NO ALIASING ?

$$\Downarrow$$

$$w_m = \frac{3\pi}{7}$$

$$w_s > 2w_m \Rightarrow \frac{2\pi}{N_s} > 2\cdot\frac{3\pi}{7} \Rightarrow N_s < \frac{7}{3} \Rightarrow \underline{N_s = 2}$$



¿EFICIENCIA?

PROBLEMA 7.16

① - $x[n]$ REAL

② - $X(e^{j\omega}) \neq 0$, $0 < \omega < \pi$ ¿$x[n]$?

③ . $x[n] \cdot \sum_{k=-\infty}^{\infty} \delta[n-2k] = \delta[n]$

$$x'[n] = \frac{\operatorname{sen} \frac{\pi}{2} n}{\pi n} \quad \text{SATISFACE 2 CONDICIONES}$$

---

① $\Rightarrow$ $|X(e^{j\omega})|$ PAR, $\sphericalangle X(e^{j\omega})$ IMPAR

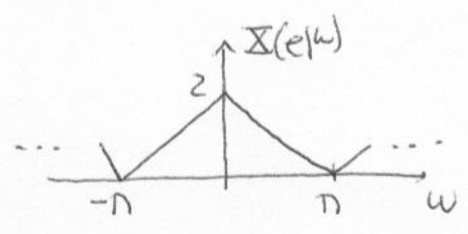
② $\Rightarrow$ "LIMITADA" EN BANDA POR π $\Rightarrow$ NO ES POSIBLE MUESTREARLA SIN SOLAPE

③ $p[n] = \sum_{k=-\infty}^{\infty} \delta[n-2k] \xrightarrow{\text{DTFT}} P(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \pi\, \delta(\omega - k\pi)$

$$\left.\begin{array}{l} x[n]\cdot p[n] \xrightarrow{\text{DTFT}} \frac{1}{2\pi} \cdot X(e^{j\omega}) \circledast P(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\omega}) + X(e^{j(\omega-\pi)})\right] \\ \delta[n] \xrightarrow{\text{DTFT}} 1 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \quad X(e^{j\omega}) + X(e^{j(\omega-\pi)}) = 2$$

UNA POSIBLE SOLUCIÓN:



PROBLEMA 7.17

FILTRO DE BANDA ELIMINADA:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \pi/4 \text{ y } |\omega| \geq 3\pi/4 \\ 0, & \text{RESTO} \end{cases} \text{, PERIÓDICO } 2\pi \longrightarrow h[n]$$

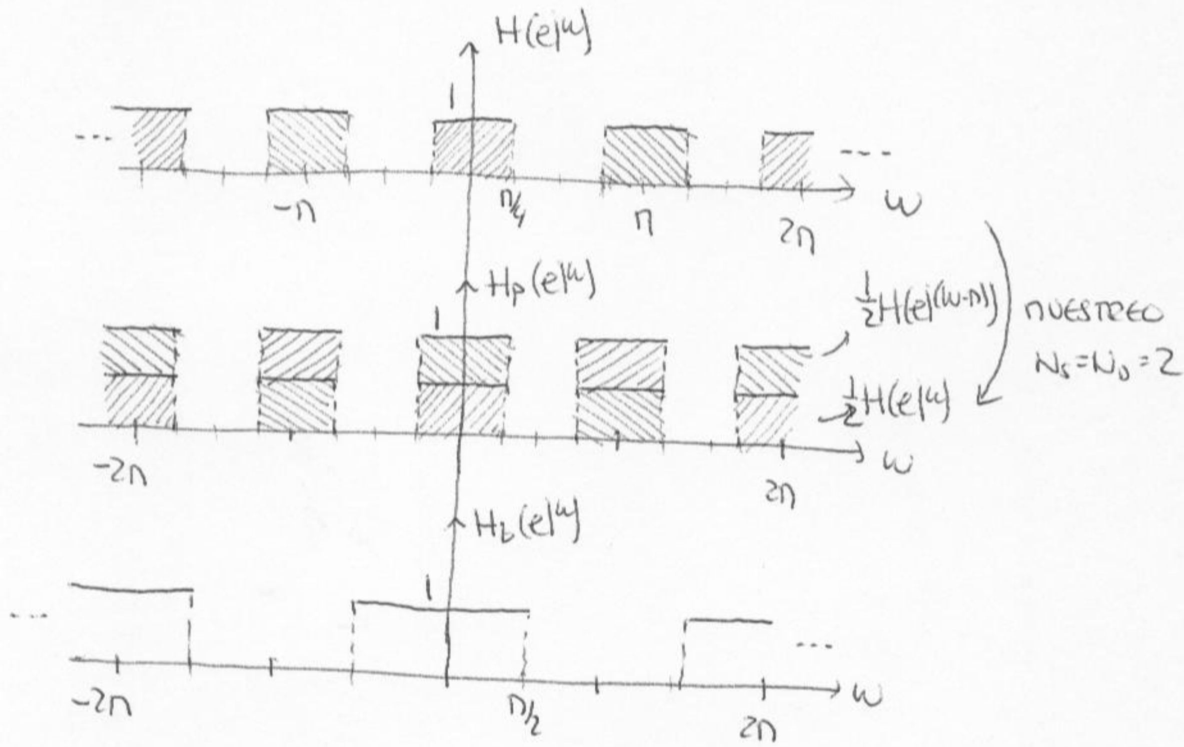
$h[2n] \longrightarrow$ ¿ $H'(j\omega)$?

$\Downarrow$

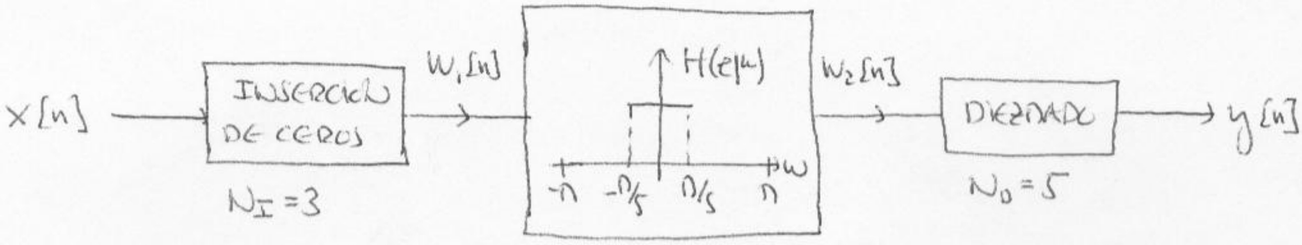
DIEZMADO POR $N_D=2$:

$$H_p(e^{j\omega}) = \frac{1}{2}\left\{H(e^{j\omega}) + H(e^{j(\omega-\pi)})\right\}$$

$$H_b(e^{j\omega}) = H_p(e^{j\omega/2}) = H'(e^{j\omega})$$



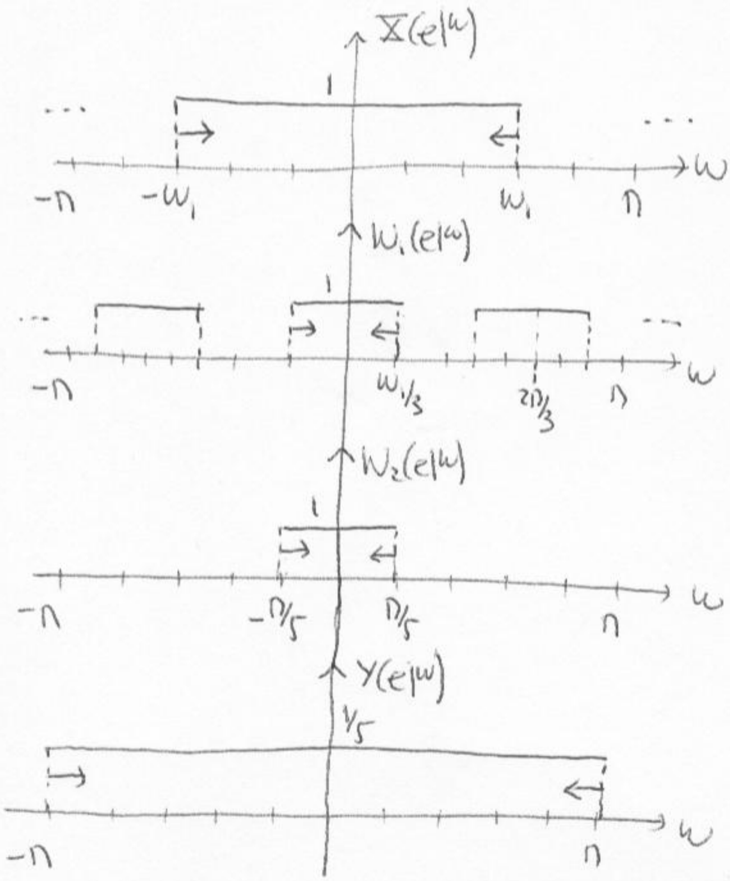
PROBLEMA 7.19



$$x[n] = \frac{\operatorname{sen} w_1 n}{\pi n} \quad , \quad ¿\, y[n] \,?$$

a) $w_1 \leq \frac{3\pi}{5}$ b) $w_1 > \frac{3\pi}{5}$

$$x[n] \longrightarrow X(e^{jw}) = u(w + w_1) - u(w - w_1)$$



$$W_2[n] = \frac{\operatorname{sen} \frac{w_1}{3} n}{\pi n} \Rightarrow$$

$$\Rightarrow y[n] = w_2[5n] = \frac{\operatorname{sen} \frac{5w_1}{3} n}{5\pi n}$$

MISMA FIGURA PERO:

$$\frac{3\pi}{5} < w_1 < \pi \Rightarrow$$

$$\Rightarrow W_2[n] = \frac{\operatorname{sen} \frac{\pi}{5} n}{\pi n} \Rightarrow$$

$$\Rightarrow y[n] = \frac{\operatorname{sen} \pi n}{5\pi n} = \operatorname{sinc}(n) =$$

$$= \frac{1}{5} \delta[n]$$