Comparison tests for improper integrals (of functions of one variable)

(1) Let $f, g : [a, \infty) \to \mathbb{R}$ be integrable functions on $[a, z]$ for all $z > a$.

(a) If $0 \le f \le g$ on $[a, \infty)$ and the improper integral $\int_a^\infty g$ converges, then the improper integral $\int_a^\infty f$ converges.

(b) If $0 \le f \le g$ on $[a, \infty)$ and the improper integral $\int_a^\infty f$ diverges, then the improper integral $\int_a^\infty g$ diverges.

(c) If f and g are non-negative and continuous functions on $[a, \infty)$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$ and $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges.

(d) If f and g are non-negative and continuous functions on $[a, \infty)$, $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$ and $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

(e) If f and g are nonnegative and continuous functions on $[a, \infty)$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} \neq 0$ (and it is finite), then $\int_a^\infty f$ converges if and only if $\int_a^\infty g$ converges.

(2) Let $f, g : (a, b] \to \mathbb{R}$ be integrable functions on $(z, b]$ for all $a < z < b$.

(a) If $0 \le f \le g$ on $(a, b]$ and the improper integral $\int_a^b g$ converges, then the improper integral $\int_a^b f$ converges.

(b) If $0 \le f \le g$ on $(a, b]$ and the improper integral $\int_a^b f$ diverges, then the improper integral $\int_a^b g$ diverges.

(c) If f and g are non-negative functions on $(a, b]$, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g$ converges, then $\int_a^b f$ converges.

(d) If f and g are non-negative functions on $(a, b]$, $\lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b f$ diverges, then $\int_a^b g$ diverges.

(e) If f and g are non-negative functions on $(a, b]$ and $\lim_{x\to a^+} \frac{f(x)}{g(x)} \neq 0$ (and it is finite), then $\int_a^b f$ converges if and only if $\int_a^b g$ converges.

(3) (a) Let $f : [a, \infty) \to \mathbb{R}$ be an integrable function on $[a, z]$ for all $a < z$. If the improper integral $\int_a^\infty |f|$ converges, then the improper integral $\int_a^\infty f$ converges.

(b) Let $f : (a, b] \to \mathbb{R}$ be an integrable function on $[z, b]$ for all $a < z < b$. If the improper integral $\int_a^b |f|$ converges, then the improper integral $\int_a^b f$ converges.

(c) **Remark.** The converse of (a) and (b) do not hold: The function $f(x) = \frac{\sin x}{x}$, for $x \in (0, \infty)$, is integrable on $(0, z)$ for all $z > 0$ and $\int_0^\infty \frac{\sin x}{x}\, dx$ converges. However, $\int_0^\infty \frac{|\sin x|}{x}\, dx$ diverges.