Extra problems

Mathematics Methods for Bioengineering

January 30, 2019

Problems

1. Prove that the planes given by equations $Ax + By + Cz + D_1 = 0$ y $Ax + By + Cz + D_2 = 0$ with $D_1 \neq D_2$ are parallel, and that the distance between them is:

$$\frac{|D_1 - D_2|}{\sqrt{A^2 + B^2 + C^2}}$$

   Note: One way of computing the distance is using the projection over the normal of any vector who links two points on the planes.

2. If a particle with mass m moves with velocity $\mathbf{v}$, his *moment* is $\mathbf{p} = m\mathbf{v}$. In a game of marbles, one marble with mass 2 grams (g) moves with speed 2 (m/s), and hits two other marbles with mass 1 g each one and stops. One of the marbles moved with speed 3 m/s forming an angle of 45° with the direction of the big. Assuming that the total moment before and after the collision is the same (according to the law of movement conservation), with what speed and angle did the second marble move?

3. Check that the inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{is} \quad \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

4. Find the volume of the parallelepiped generated by the vectors $(1, 0, 1)$, $(1, 1, 1)$ y $(-3, 2, 0)$.

5. Sketch the graph of $f(x, y) = \sqrt{(100 - x^2 - y^2)}$. Help yourself using contour/level curves and the intersection of the graph with the planes $x = c$ e $y = c$, $c =$constant.

6. A cistern that has the shape of a circular cylinder with a radius of 3 m and a height of 5 m is filled to a half with liquid and lies on its side. Describe the empty space inside the cistern by choosing a cylindrical coordinate system.

   Observation: An unbounded, lying down cylinder of radius r_0, can be written in Cartesian coordinates as $x^2 + (z - r_0)^2 = r_0^2$.

7. Compute the following limit

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = 0$$

   Note: Find an appropriate coordinate system.

8. OPTIONAL: Let $f : A \in \mathbb{R}^n \to \mathbb{R}^m$ that, for certain constant K satisfies $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq K\|\mathbf{x}-\mathbf{y}\|$ for all $\mathbf{x}$ and $\mathbf{y}$ en A. Show that f is continous with the ε, δ notation (the functions who verifies the previous property are called **Lipschitz**). Give an example function.

9. Find a equation of the tangent plane of the surface $z = x^2 + y^3$ in $(3, 1, 10)$.

10. Show that the graph of $f(x, y) = x^2 + y^2$ and $g(x, y) = -x^2 - y^2 + xy^3$ are tangent at the origin.

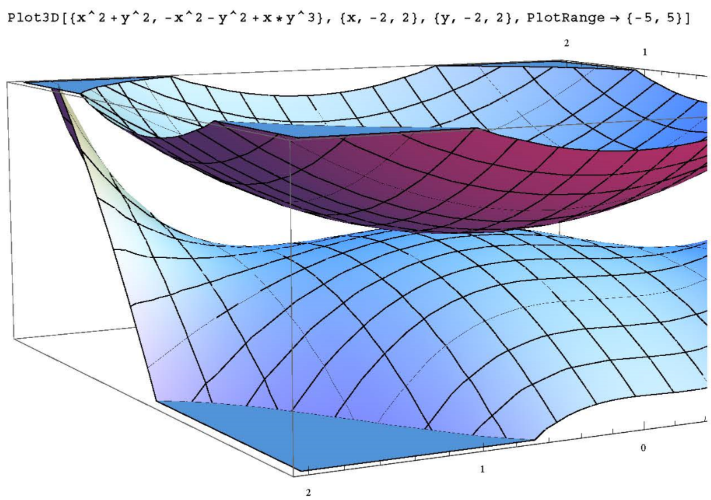

Figure 1: Surface graph of functions in exercises 11.

11. Compute the Jacobian matrix of $f(x, y) = (xye^{xy}, x \sin y, 5xy^2)$.

12. Suppose a duck swims over the circle $x = \cos t, y = \sin t$, and that the temperature of the water is given by the formula $T = x^2 e^y - xy^3$. Find $\frac{dT}{dt}$, the instant rate of temperature change that the duck feels:

    (a) By means of the chain rule.

    (b) Expressing T as function of t and calculating the derivative.

13. This exercise shows that the chain rule can not be applied if f is not differentiable. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

    See that:

    (a) The partial derivatives exists at $(0, 0)$.

    (b) If $\mathbf{g}(t) = (at, bt)$ for some constants a, b, then $f \circ \mathbf{g}$ is derivable but the chain rule is false: $(f \circ \mathbf{g})'(0) \neq \nabla f(0, 0) \cdot \mathbf{g}'(0)$.

14. Suppose that a mountain is shaped like an elliptic paraboloid $z = c - ax^2 - by^2$, where a, b, c are positive constants, x is the east-west coordinate, y north-south and z is the altitude above sea level.

(a) At the point $(1, 1)$, in which direction does the altitude grow fastest? If a ball is released at that point, in which direction will it begin to roll ?

(b) An engineer wants to build a railway that goes up the mountain. If he buil it directly to the top it would have too much slope for the power of the machine. In the point $(1, 1)$, in what directions could the road be laid so that it climbs with a slope of 3%? Solve it for the specific case $a = 1,\ b = 0.5$.

Note: A slope of 3% means going up with an inclination angle whose tangent is 0.03. That is, every 100 steps that is advanced horizontally is climbed 3 vertically.

Note: There are two possible directions for section b).