Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 2.1 Higher order derivatives. Hessian Matrix

Problem 1. Consider the function

$$f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0).\end{cases}$$

i) Compute the partial derivatives outside the origin.

ii) Show that $\dfrac{\partial f}{\partial x}(0,0)=\dfrac{\partial f}{\partial y}(0,0)=0$.

iii) Show that $\dfrac{\partial^2 f}{\partial x\partial y}(0,0)=1$, while $\dfrac{\partial^2 f}{\partial y\partial x}(0,0)=-1$. Explain these results.

Solution: $i)$ $\dfrac{\partial f}{\partial x}=\dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}$, $\dfrac{\partial f}{\partial y}=-\dfrac{xy^4+4x^3y^2-x^5}{(x^2+y^2)^2}$; $iii)$ f is not of class C^2 on any neighbourhood of the origin.

Problem 2. Verify that

$$\frac{\partial^3 f}{\partial x\partial y\partial z}=\frac{\partial^3 f}{\partial y\partial z\partial x},$$

for the function $f(x,y,z)=ze^{xy}+yz^3x^2$. Then, Schwarz's Theorem is satisfied and f is of class $\mathcal{C}^3$.

Problem 3. Compute the Hessian matrix for the functions:

i) $f(x,y)=\log(x^2+y^3)$;

ii) $f(x,y)=x^2\cos y+y^2\sin x$;

iii) $f(x,y)=x^y$;

iv) $f(x,y)=\arctan(xy)$;

v) $f(x,y)=\arctan(x/y)$;

vi) $f(x,y,z)=(x+y)(y+z)(z+x)$.

Solution: $i)$ $\begin{pmatrix}\frac{2y^3-2x^2}{(x^2+y^3)^2} & -\frac{6xy^2}{(x^2+y^3)^2}\\ -\frac{6xy^2}{(x^2+y^3)^2} & \frac{6x^2y-3y^4}{(x^2+y^3)^2}\end{pmatrix}$; $ii)$ $\begin{pmatrix}2\cos y-y^2\sin x & -2x\sin y+2y\cos x\\ -2x\sin y+2y\cos x & -x^2\cos y+2\sin x\end{pmatrix}$;

$iii)$ $\begin{pmatrix}y(y-1)x^{y-2} & x^{y-1}(1+y\log x)\\ x^{y-1}(1+y\log x) & x^y\log^2 x\end{pmatrix}$; $iv)$ $\begin{pmatrix}\frac{-2xy^3}{(1+x^2y^2)^2} & \frac{1-x^2y^2}{(1+x^2y^2)^2}\\ \frac{1-x^2y^2}{(1+x^2y^2)^2} & \frac{-2x^3y}{(1+x^2y^2)^2}\end{pmatrix}$;

$v)$ $\begin{pmatrix}\frac{-2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2}\\ \frac{x^2-y^2}{(x^2+y^2)^2} & \frac{2xy}{(x^2+y^2)^2}\end{pmatrix}$; $vi)$ $\begin{pmatrix}2y+2z & 2x+2y+2z & 2x+2y+2z\\ 2x+2y+2z & 2x+2z & 2x+2y+2z\\ 2x+2y+2z & 2x+2y+2z & 2x+2y\end{pmatrix}$.