TIME DOMAIN SIGNALS

LINEAR SYSTEMS WITH CIRCUIT APPLICATIONS

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Biomedical Engineering Degree

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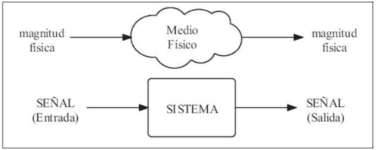
Introduction

Signals and Systems (I)



- A physical phenomenon is the variation, transformation of a given physical magnitude into another, due to the interaction with a physical environment.

- The mathematical model that represents the transformation of these physical magnitudes is called **signal** and the mathematical model that represents the effect of the physical environment is called **system**.
- Usually, simply measuring input and output magnitudes we are able to know the effect of the physical envorinment, and therefore, to define mathematically the systems as:

$$y(t) = F(x(t))$$

- The aim of the *Signal and Systems Theory* is to represent mathematically the physical phenomena by defining the systems and determining the input and output signals.

Introduction

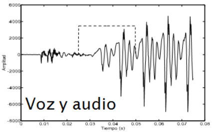
Signals and Systems (II)

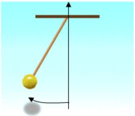
- **Signal** is a mathematical function of one or more independent variables, which contains information about a physical magnitude. As a mathematical function is usually represented as: $u = f(t)$, being time the independent variable. Some examples are:
  - Signal *one-dimensional*. Involving *one* single independent variable, $u = f(t)$: speech recordings, stock market series, *electrocardiogram, ECG*.
  - Signal *two-dimensional*. Involving *two* independent variables, $u = f(x, y)$: gray-scale images.
  - Signal *three-dimensional*. Involving *three* independent variables, $u = f(x, y, t)$: video.
- **System** is the mathematical abstraction that represent a device (equipment) that transform an input signal (the inpunt signal cause the system to respond) into an output signal (is the response of the system)
- Signals are mathematical inputs acting as a: input, output or internal signal, that the systems process or produce.
- For example, in a electric circuit, voltages and currents through the elements of the circuit, as a function of time, are **signals**; whereas the whole circuit is the **system** itself.

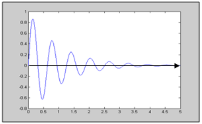
Introduction

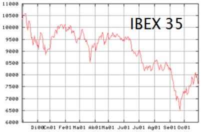
Examples of signals and systems (I)

- Signal and systems concepts arise in many fields: communications, aeronautics, circuit design, **biomedical engineering**, power energy...
- Signals are used to represent physical magnitudes: *speech signal* represents acoustic pressure variations, ECG signal represents myocardial cellular electric currents, or the digital signal used in radio communications.
- **Systems** are used to represent the means that process, distort or integrate signals: e.g. microphone, muscles in human body, atomosphere...



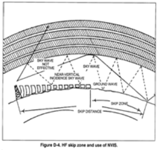


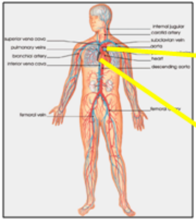
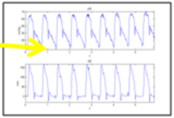
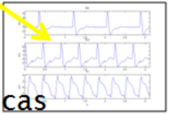




Introduction

Signals and Systems examples (II)

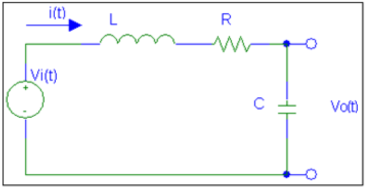



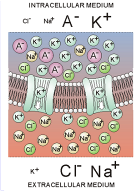





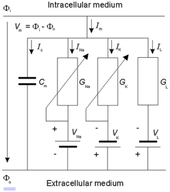
Señales biomédicas

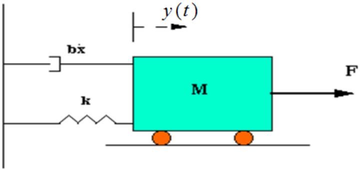
Introduction

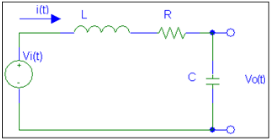
Signals and Systems examples (III)











Introduction

Continuous and Discrete time (I)

- **Continuous time signals**: the independent variable is continuous (real in math sense), and thus these signals are defined for a continuum of values.

$$x(t), t \in \mathbb{R}$$

- **Discrete time signals**: they are defined only at discrete times, and consequently, for these signals, the independent variable takes on only a discrete set of values (integers in math sense). Sometime, they are called **discrete time sequences**, or **sequences**, for short.

$$x[n], n \in \mathbb{Z}$$

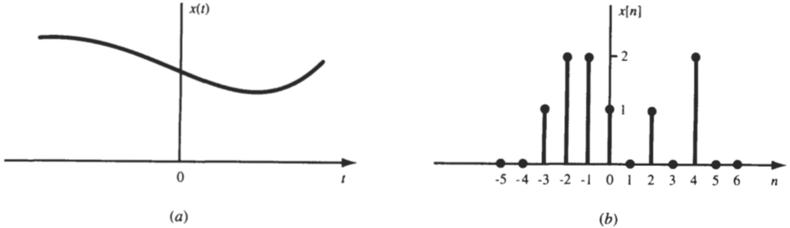


Fig. 1-1 Graphical representation of (a) continuous-time and (b) discrete-time signals.

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Properties of signals

Real and Complex signals (I)

- A signal, $x(t)$ is real if its value is a real number. $x(t) : \mathbb{R} \rightarrow \mathbb{R}$.
- For example, $x(t) = t^2$, or $x(t) = 3$.
- A signal, $z(t)$ is complex if its value is a complex number. $z(t) : \mathbb{R} \rightarrow \mathbb{C}$.
- For example, $x(t) = cos(2t) + jsen(5t)$, or $x(t) = \sqrt{t}$.
- To graphically represent a real signal we can use one graph. However, to represent a complex signal we'll need two separate graphs.
- Remember that a complex signal can be in either *rectangular form* or polar form:
    - Rectangular form: real and imaginary part:

$$x\left(t\right) = \Re\{x\left(t\right)\} + j\Im\{x\left(t\right)\} = a\left(t\right) + jb\left(t\right)$$

    - Magnitude and argument (modulus and phase)):

$$x\left(t\right) = \left|x\left(t\right)\right| e^{j\angle\{x(t)\}} = \left|x\left(t\right)\right|_{\angle\{x(t)\}}$$

- Remember: It is very important to be (very) familiar with Euler's Identity and manipulations with complex numbers.

$$\rho e^{j\phi} = \rho\cos(\phi) + j\rho\sin(\phi)$$

Properties of signals

Real and Complex signals (II)

- Complex conjugate of a signal:

$$x^*(t) = \Re\{x(t)\} - j\Im\{x(t)\} = a(t) - jb(t)$$

- Real and imaginary parts can be obtained as:

$$\Re\{x(t)\} = \frac{1}{2}\left[x(t) + x^*(t)\right]; \quad \Im\{x(t)\} = \frac{1}{2j}\left[x(t) - x^*(t)\right]$$

- The magnitude and argument can be obtained as:

$$|x(t)|^2 = x(t) \cdot x^*(t) = (\Re\{x(t)\})^2 + (\Im\{x(t)\})^2$$

$$\angle\{x(t)\} = \operatorname{arc\,tg}\frac{\Im\{x(t)\}}{\Re\{x(t)\}}$$

Questions

2 (*)Compute and represent real and imaginary part, and magnitude and argument:

- $x_1(t) = \cos(\pi t) + j\,\mathrm{sen}(\pi t)$.
- $x_2(t) = \sqrt{t}$.
- $x_3(t) = e^{-2t}e^{-j2t}$.

Properties of signals

Symmetry in real signals

- A real signal is **even** if it is identical with its reflection about the origin, i.e. $x(t) = x(-t)$.
- For example, $x(t) = t^2$, or $x(t) = \cos(\pi t)$, are even signals.
- A real signal is **odd** if it is antisymmetric with its reflection about the origin, i.e., $x(t) = -x(-t)$
- For example $x(t) = t$, $x(t) = \sin(t)$, are odd signals.
- There are signals that have no symmetry, but any signal can be broken into a sum of two signals, one of which is even and one of which is odd.

$$x(t) = x_e(t) + x_o(t)$$

- Even and odd parts can be obtained as:

$$x_e(t) = \frac{1}{2}[x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2}[x(t) - x(-t)]$$

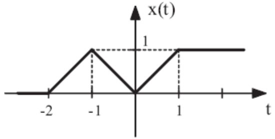
Properties of signlas

Questions

3 (*)Study the symmetry of:
   1. $x(t) = sen(\pi t)$.
   2. $y(t) = \cos(2\pi t)$.
   3. $z(t) = e^{-\alpha t}$, with $\alpha \in \mathbb{R}$.

4 (*)Find the even and odd components of the previous signlas

5 (*)Find the even and odd component of the signal in Figure.



Propoerties of signals

Symmetry in complex signals

- **Hermitian** symmetry. A complex signal is hermitian when is conjugate symmetric with its reflection about the origin, i.e., $x(t) = x^*(-t)$.
- For example, $x(t) = e^{jt}$ and $x(t) = j\sin(t)$ are hermitian.
- Additionally:

$$\text{If } x(t) \text{ is hermitian} \Rightarrow \Re\{x(t)\} \text{ is even } \Im\{x(t)\} \text{ is odd.}$$
$$\text{If } x(t) \text{ is hermitian} \Rightarrow |x(t)| \text{ is even and } \angle\{x(t)\} \text{ is odd.}$$

- **Antihermitian** sysmemtry. A complex signal is **antihermitian** when is antisymmetryc whith its reflection about the origin, i.e., $x(t) = -x^*(-t)$.
- For example, $x(t) = t + j$ and $x(t) = j\cos(t)$ are antihermitian.
- Every complex signal has two components: a **hermitian part** and an **antihermitian part**, that is,

$$x(t) = x_h(t) + x_a(t)$$

- Hermitian and antihermitian components can be computed as:

$$x_h(t) = \frac{1}{2}[x(t) + x^*(-t)]; \quad x_a(t) = \frac{1}{2}[x(t) - x^*(-t)]$$

Properties of Signals

Questions

6 Find the hermitian and anti hermitian components:
   1. $x(t) = \cos(\omega_0 t) + j\sin(\omega_0 t)$.
   2. $y(t) = e^{-2t}e^{5jt}$.

7 Show that:
   1. if $x(t)$ is hermitian $\Rightarrow \Re\{x(t)\}$ is even and $\Im\{x(t)\}$ is odd.
   2. if $x(t)$ is hermitian $\Rightarrow |x(t)|$ is even and $\angle\{x(t)\}$ is odd.

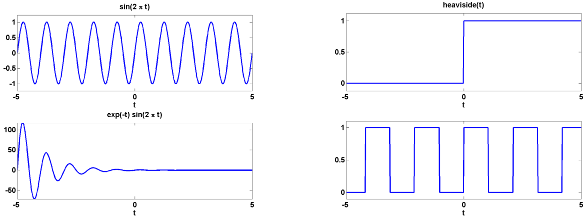
8 Study the symmetries $\Re\{x(t)\}$, $\Im\{x(t)\}$, $|x(t)|$ y $\angle\{x(t)\}$, when $x(t)$ is a complex antihermitian signal.

Properties of signals

Periodicity

- A signal is said to be **periodic** if we can find a constant time interval T, so taht the values of the signal are repetead every T. This time interval, T is called **period**.

$$x(t) \text{ is periodic} \Leftrightarrow \exists\, T > 0,\, T \in \mathbb{R} \text{ so that } x(t) = x(t+T)\ \forall t$$



Fundamental period

- If $x(t)$ is periodic with period T, it is also periodic with periods $2T, 3T, \ldots$.
- We call *fundamental period*, T_0, to the smallest value of T for which the equation $x(t) = x(t+T)$ holds.

Properties of Signals

Example: Periodic Signals

- We want to find out if $x(t) = \cos(\frac{2\pi}{5}t)$ is a periodic signal. ¿$\exists T$ such that $x(t) = x(t+T)\ \forall t$?
- To answer, we have to find $x(t+T)$:

$$x(t+T) = \cos\left(\frac{2\pi}{5}(t+T)\right) = \cos\left(\frac{2\pi}{5}t + \frac{2\pi}{5}T\right) =$$
$$= \cos\left(\frac{2\pi}{5}t\right)\cos\left(\frac{2\pi}{5}T\right) - \operatorname{sen}\left(\frac{2\pi}{5}t\right)\operatorname{sen}\left(\frac{2\pi}{5}T\right)$$

- Now, we need to choose an appropriate T, so that the previous signal is equivalent to $x(t)$, therefore:

$$\cos\left(\frac{2\pi}{5}T\right) = 1, \quad \operatorname{sen}\left(\frac{2\pi}{5}T\right) = 0$$

  Thereby, every $T = 5\ k$, with $k = 1, 2, \ldots$ is a valid periodof the signal $x(t)$. Thus, we can assure that the signal is periodic.

Properties of Signals

Questions

9 Show that if $x_1(t)$ y $x_2(t)$ are periodic signlas with period T_0, then the signal $y(t) = x_1(t) + x_2(t)$ is also periodic. ¿Which is its period?

10 Show that if $x_1(t) = x_1(t + T_1)$ y $x_2(t) = x(t + T_2)$ holds, then the signal $y(t) = x_1(t) + x_2(t)$ is periodic. ¿Which is its period?

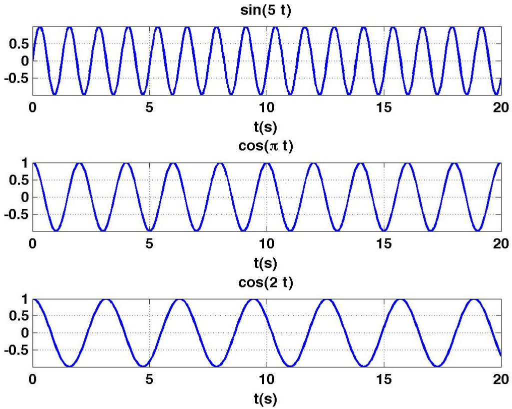
Questions

11 (*)Study if the following signals are prediocis, if so, find their periods.

- $x_1(t) = \cos(\omega_0 t)$
- $x_2(t) = \sin\left(\omega_0 t + \frac{1}{2}\right)$
- $x_3(t) = e^{j\omega_0 t}$
- $x_4(t) = \cos(10\pi t)$
- $x_5(t) = \sin(10\pi t) + \cos(20\pi t)$
- $x_6(t) = \sin(10\pi t) + \cos(20t)$
- $x_7(t) = \sin(10\pi t)\cos(20\pi t)$

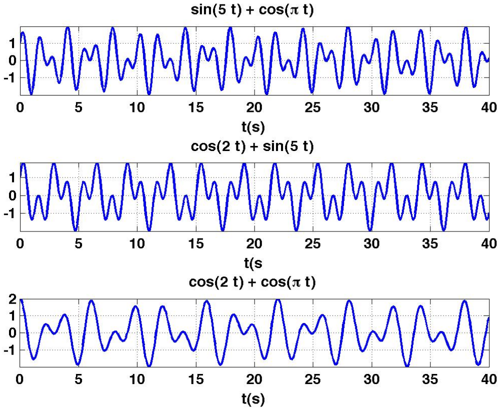
Properties of Signals

Example: periodic and non periodic signals



Properties of Signals

Example: periodic and non periodic signals



Properties of Signals

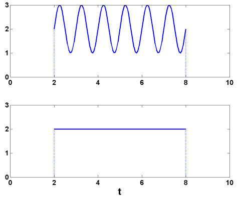
Average value of a signal in an interval

- We want to characterize a signal with different measurements within an interval, such as: avera value, average power or energy.
- Any of the following measurements can be computed on an interval or on the whole signal.
- To define a time interval we need to specify the begin (t_B) and the end (t_E) of the interval: (t_B, t_E). This can also be defined as an interval centered at t_0 with duration T, or the interval $[T, t_0]$.

Average value in a finite time interval (I)

$$\text{area} = \int_{t_i}^{t_f} x(t)dt = m \cdot T$$

$$m = \frac{\text{area}}{T}$$



Properties of Signals

Average value in a finite time interval (II)

- The average value of a signal, also known as *DC level (Direc Current)* in a finite time interval, can be computed as:

$$\langle x(t)\rangle_{[T,t_0]} = \frac{1}{T}\int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} x(\tau)d\tau; \quad \langle x(t)\rangle_{(t_i,t_f)} = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} x(\tau)d\tau$$

Total Average Value

- The Total Average Value of signal $x(t)$ is defined as::

$$m_\infty = \langle x(t)\rangle = lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} x(\tau)d\tau\right\}$$

- **Total Average Value for periodic Signals**. Since in periodic signals $x(t) = x(t+T_0)$, the average value in a period would be the same as the total average value $(-\infty,\infty)$, therefore, is easier to compute:

$$m_\infty = \langle x(t)\rangle = \frac{1}{T_0}\int_{\langle T_0\rangle} x(\tau)d\tau$$

Properties of Signals

Questions

12 (*)Find the average value of the following signals:

- $x_1(t) = e^{-t}$, calcular $\langle x_1(t) \rangle_{(2,3)}$.
- $x_2(t) = t^2$, calcular $\langle x_2(t) \rangle_{(1,3)}$.
- $x_3(t) = |\sin(t)|$, calcular $\langle x_3(t) \rangle$.
- $x_4(t) = \cos(\frac{\pi t}{2})$, calcular $\langle x_4(t) \rangle$.
- $x_5(t) = u(t)$, calcular $\langle x_5(t) \rangle_{(-1,3)}$
- $x_6(t) = u(t)$, calcular $\langle x_6(t) \rangle$
- $x_7(t) = e^{j\left(5\pi t - \frac{1}{2}\right)}$, calcular $\langle x_7(t) \rangle_{(-1,3)}$.
- $x_8(t) = e^{j\left(5\pi t - \frac{1}{2}\right)}$, calcular $\langle x_8(t) \rangle$.

Properties of Signals

Power and Energy in Signals

- Power and energy are concepts used in physics, for example, in circuits. We are going to define, using the analogy, abstract concepts of **Power** and **Energy** for signals.
- We are going to define the power consumed by a reference resistor $R = 1\Omega$. We can think that $x(t)$ is either $v(t)$ or $i(t)$. The instantaneous power $p(t)$ is defines as:

$$p(t) = |v(t)|^2/R = |i(t)|^2 R$$

- The total energy E and power P consumption is:

$$E = \int_{-\infty}^{\infty} i^2(t)dt \quad joules$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)dt \quad watts$$

Properties of Signals

Power of a Signal

- The instantaneous power os a signal is defined as the square magnitude of the signalLa potencia instantánea de una señal se define como el módulo al

$$p_x\left(t\right) = \left|x\left(t\right)\right|^2$$

- The avearge power in a given interval of length T, (t_1, t_2):

$$P_T = \langle p_x(t) \rangle_{(t_1, t_2)} = \frac{1}{T}\int_{t_1}^{t_2} p_x(\tau) d\tau = \frac{1}{T}\int_{t_i}^{t_f} |x(\tau)|^2 d\tau$$

- The total average power:

$$P_\infty = \langle p_x(t) \rangle = \lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} p_x(\tau) d\tau\right\} = \lim_{T\to\infty}\left\{\frac{1}{2T}\int_{-T}^{T} |x(\tau)|^2 d\tau\right\}$$

.

- If the signal is periodic, period T_0, the average power is computed as:

$$P_\infty = \langle p_x(t) \rangle = \frac{1}{T_0}\int_{\langle T_0 \rangle} p_x(\tau) d\tau = \frac{1}{T_0}\int_{\langle T_0 \rangle} |x(\tau)|^2 d\tau$$

Properties of Signals

Energy of a signal

- The energy in circuits can be computed as

$$p_x(t) = \frac{dw(t)}{dt} \Rightarrow w(t) = \int_{-\infty}^{t} p_x(\tau)d\tau$$

- Therefore, the total energy can be computed as:

$$E_\infty = \lim_{T\to\infty} w(t) = \int_{-\infty}^{+\infty} |x(\tau)|^2 d\tau$$

.

Classification of signals regarding energy and power

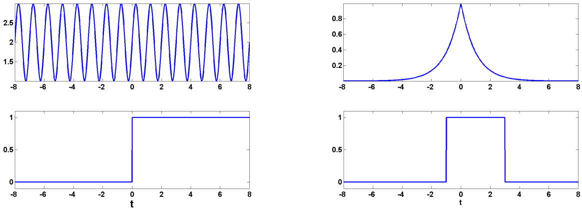
- **Signals with finite energyy** $0 < E_\infty < \infty$.
- For example, signals with limited duration.
- **Signals with finite average power** $0 < P_\infty < \infty$.
- For example, periodic signals.

Questions

13 Show that any signal with finite energy has zero average power, and also that any signal with finite average power has infinite energy.

Properties of Signals

Examples: Signals with finite average power and signals with finite energy.



Questiones

14 (*)Find the average power and eneryg for each of the following signals.

- $x_1(t) = u(t)$
- $x_2(t) = e^{-2t} \cdot u(t)$
- $x_3(t) = e^{j(2t + \frac{\pi}{4})}$
- $x_4(t) = \cos(t)$
- $x_5(t) = (\frac{1}{2})^t \cdot u(t)$
- $x_6(t) = (3 + 2j)u(t)$

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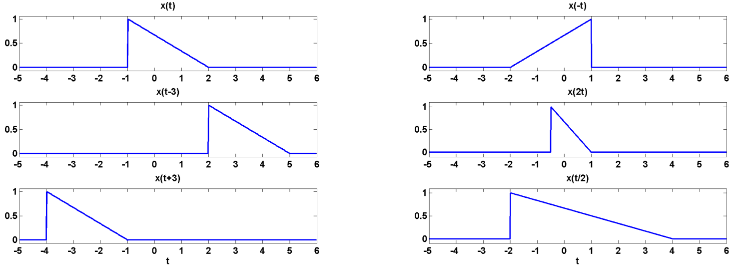
Transformations of signals

Transformations of the Independent Variable

- There are 3 basic transformations of the independent variable of a signals:
  - Shifting: $y(t) = x(t \pm a)$, with $a > 0$.
  - Scaling: $y(t) = x(at)$, with $a > 0$.
  - Time-reversal: $y(t) = x(-t)$.



Transformations of signals

Shift

- Let's be $a \in \mathbb{R}^+$. the result of add or subtract a to the independent variable is a new shifted signal, $y(t) = x(t \pm a)$.
  - $y_1(t) = x(t - a)$ is a delayed version of $x(t)$ shifted to the right.
  - $y_2(t) = x(t + a)$ is an advanced version of $x(t)$ shifted to the left.
- If $x(t)$ is a song of $\ldots$, $x(t - a)$ means you are going to play the song some time ahead, whereas $x(t + a)$ means that you have already played the song.

Time reversal

- Multiply by -1 the independent variable (sign change) results in a new signal $y(t) = x(-t)$, which is the same as the original but reflected around $t = 0$. This is the song played backward.

Scaling

- Let be $a \in \mathbb{R}^+$. Signal $y_1(t) = x(at)$, when $a > 1$, is an accelerated version of $x(t)$, while, with $a < 1$, is a slower version.
- For example, if $x(t)$ is a song, $x(2t)$ is the song played at twice the speed, and $x(t/2)$ is played at half-speed.
- Note that scaling in continuous time does not implyt lost of information. We can always recover original signal from a scaling one, using a new scaling on the transformed signal $a' = 1/a$.

Transformation of Signals

Practical advices

- Whener you have several transformations, it is easier (commonly) to start by the time shifting.
- You are only transforming the independent variable.
- It is always a good advice to use intermediate signals, plotting them and keeping the analytical expressions.
- At the end, evaluate always the result in known values of the independent variable

$$y(t) = x(\alpha t + \beta) \Rightarrow y(t)\Big|_{t^*}$$

Where t^* is an easy value where check the transformation.

Example

- we want $v(t) = x(at + b)$.
  - Start with shifting:

$$z(t) = x(t + b)$$
$$s(t) = z(at) = x(at + b) = v(t) \text{ (OK)}$$

  - Start with scaling:

$$z(t) = x(at)$$
$$r(t) = z(t + b) = x(at + ab) \neq v(t) \text{ (!!)}$$

Transformation of signals

Questions

15 How the order affects $v(t) = x(-t+b)$? ¿How the order affects$x(-at)$?

16 Given 15, which is the order to follow when we have transformation of the independent variable?

Transformations of the dependent variable

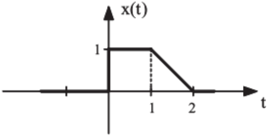
- Let be a a scalar, then $y(t) = ax(t)$ is an amplified $(a > 1)$ or reduced version $(0 < a < 1)$ of the signal $x(t)$ $\forall t$.
- $y(t) = x(t) + a$ is a new signal that just adds the quantity a to every valued of the signal $s(t)$
- $y(t) = -x(t)$ is just change the sign of every value of the signal $x[t]$.

Transformation on Signals

Questions

17 (*)Let be $x(t)$ the signal in the Figure. Sketch and label properly the following transformations:

- $y_1(t) = x(-t+1)$
- $y_2(t) = x(2t+3)$
- $y_3(t) = x\left(\frac{3}{2}t+1\right)$
- $y_4(t) = -2 \cdot x\left(-\frac{t}{4}+1\right)+3$



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Basic Signals

Why we need basic signals

- We are going to study different simple and basic signals that are going to be use as **building blocks** to compose more complex signals.
- Why is that?
  - They are simple signals, so their properties can be studied easily.
  - Almost any signal can be composed as a linear combination of these building blocks.
  - The transformation of a simple signal by a system is easy to study.

Basic Signals

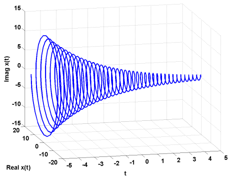
Continuous-time complex exponential (I)

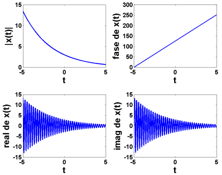
- The more general expression for a continuous-time complex exponential is:

$$x(t) = C \cdot e^{at}$$

where $C, a \in \mathbb{C}$, dados por $C = |C| \cdot e^{j\phi}$ y $a = \sigma + j\Omega$. Therefore,

$$x(t) = |C|e^{j\phi}e^{\sigma t}e^{j\Omega t} = |C|e^{\sigma t}e^{j(\Omega t+\phi)} = |C|e^{\sigma t}\left(\cos(\Omega t + \phi) + j\sin(\Omega t + \phi)\right)$$





- Depending upon the values of these parameters the complex exponential can exhibit different characteristics.

Question

18 Find the magnitude, phase, and real and imaginary partos of $x(t)$, givne by the continuous-time complex exponential. What is the magnitude and phase of $x(t)$ at $t = 0$, and at $t = 1$, and at $t = -\pi$?

Basic Signals

Continuous-time complex exponential (II)

- **Real exponential** when $C, a \in \mathbb{R}$. That is, $x(t) = Ce^{\sigma t}$.
  - It can be a growing exponential($\sigma > 0$) or a decaying exponential ($\sigma < 0$).
- **Purely imaginary exponentials**, when $C \in \mathbb{C}$ but $a = j\Omega$ is purely imaginary. In that case,

$$x(t) = Ce^{j\Omega t} = |C|\left(\cos(\Omega t + \phi) + j\sin(\Omega t + \phi)\right)$$

  - It is very easy to establish the relationship between complex exponentias an sinusoidal signals:

$$e^{j\theta} = \cos\theta + j\sin\theta; \quad \cos\theta = \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right); \quad \sin\theta = \frac{1}{2j}\left(e^{j\theta} - e^{-j\theta}\right)$$

    therefore

$$\cos\Omega t = \frac{1}{2}\left(e^{j\Omega t} + e^{-j\Omega t}\right); \quad \sin\Omega t = \frac{1}{2j}\left(e^{j\Omega t} - e^{-j\Omega t}\right)$$

  - In a purely imaginary exponentail (or in a sinusoidal signal for that matter) the frequency can grow infinitely.

Basic Signals

Questions

19 Let be the following sinusoidal signals $x(t) = 100\cos(400\pi t + 60^{o})$.

1. What is the maximum amplitude of the signal?
2. What is the frequency in Herz? What is the frequency in rad/sg?
3. What is the phase in radianes? What is the phase in degrees?
4. What is the period in milliseconds?
5. What is the first time, after $t = 0$, that $x = 100$?

20 Show that a purely imaginary exponential signal is always periodic.

21 (*)Find whether the following signals are periodic or not. If periodic, find the fundamental period.eriódica, especifique su periodo fundamental.

- $x_1(t) = j \cdot e^{10jt}$
- $x_2(t) = e^{(-1+j)t}$
- $x_3(t) = 2\cos(10t + 1) - \sin(4t - 1)$
- $x_4(t) = 1 + e^{j\frac{4\pi t}{7}} - e^{j\frac{2\pi t}{5}}$
- $x_5(t) = [\cos(2t - \frac{\pi}{3})]^2$

22 (*)Sketch the following signals and indicate, using the plot, whether they are periodic or not.

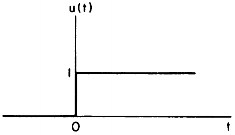
- $x_0(t) = u(t) - u(t-1)$
- $x_1(t) = \sum_{k=-1}^{2} x_0(t - 2k)$
- $x_2(t) = \sum_{k=-\infty}^{\infty} x_0(t - 2k)$

Basic Signals

Continuous-Time Unint Step

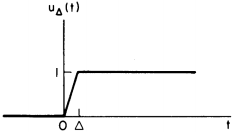
- **The continuous-time unit step** is defined as follows::

$$u(t) = \begin{cases} 0, & t < 0, \\ 1, & t > 0 \end{cases}$$



- Note that the unit step is discontinuous at $t = 0$. To solve this, we define $u(t)$ using an approximation signal:

$$u_\Delta(t) = \begin{cases} 0, & t < 0 \\ \frac{t}{\Delta}, & 0 \le t \le \Delta \\ 1, & t > \Delta \end{cases}$$



- Therefore, $u_\Delta(t)$ is a continuous approximation of the unit step and

$$u(t) = \lim_{\Delta \to 0} u_\Delta(t)$$

Basic Signals

Continuous-Time Unit Impulse

- Also knwon as **Dirac delta**:

$$\delta(t) = \begin{cases} \infty, & t = 0 \\ 0, & t \neq 0 \end{cases}$$

  but **area equal to 1**.

- We can define unit impulse as the **first derivative** of the unit step:

$$\delta(t) = \frac{du(t)}{dt}$$

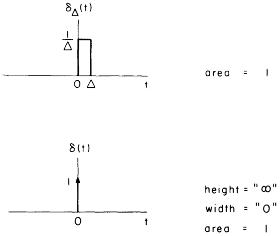
  But this arise some problems, since $u(t)$ is discontinuous at $t = 0$.

- We can use the approximation of the unit step, $u_\Delta(t)$, for which the deritave is well defined:

$$\delta_\Delta(t) = \frac{du_\Delta(t)}{dt}$$

- **Note** that $\delta_\Delta(t)$ is a short pulse, of duration Δ and with unit area for any value of Δ. As $\Delta \to 0$, $\delta_\Delta(t)$ becomes narrower and higher, maintining its unit are. Therefore, at the limit:

$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



Basic Signals

Properties of the unit impulse

1. The area under the function is 1:
$$\int_{-\infty}^{+\infty} \delta(\tau) d\tau = 1$$
2. Scaling property:
$$\delta(at) = \frac{1}{|a|}\delta(t)$$
3. Even property
$$\delta(-t) = \delta(t)$$
4. Sampling property
$$x(t)\delta(t) = x(0)\delta(t)$$
5. Sampling property (ii)
$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)$$
6. Sampling property (iii)
$$x(t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t_0 - \tau) d\tau$$
7. Therefore, any continuous-time signal can be decompose as a (infinte) linear combination of shifted and scaled unit impulses
$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t - \tau) d\tau$$

Basic Signals

Questions

27 Show, and discuss, the meaning of the properties 4. 5 and 6.

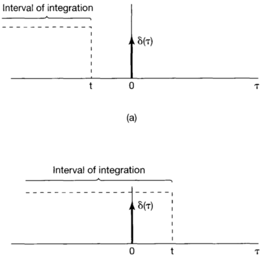
28 Show that the area under the signal $x(t) = A\delta(t)$ is equal to A. **Hint:** Use the approximation $\delta_{\triangle}(t)$.

Basic Signals

Relationship between unit step and unit impulse (I)

- The relationship between unit step and unit impulse allows to deal with derivative of discontinuities
- Running integral definition:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



Basic Signals

Relationship between unit step and unit impulse (II)

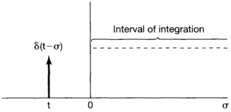
- We can use an alternative definition:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \int_{\infty}^{0} \delta(t-\sigma)(-d\sigma)$$

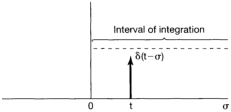
  with $\sigma = t - \tau$

- Or equivanlently:

$$u(t) = \int_{0}^{\infty} \delta(t-\sigma)d\sigma$$



(a)



Basic Signals

Relationship between unit step and unit impulse (III)

- **Derivative of discontinuites**

$$\delta(t) = \frac{du(t)}{dt} \Rightarrow u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau$$

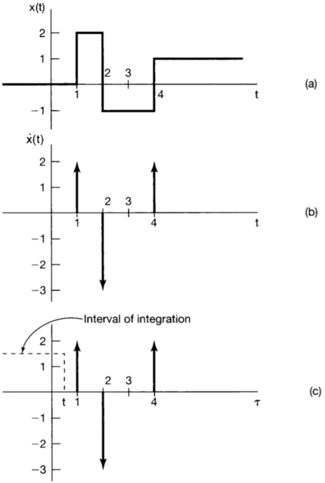


Figure 1.40 (a) The discontinuous signal $x(t)$ analyzed in Example 1.7; (b) its derivative $\dot{x}(t)$; (c) depiction of the recovery of $x(t)$ as the running integral of $\dot{x}(t)$, illustrated for a value of t between 0 and 1.

Basic Signals

Questions

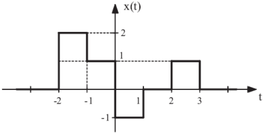
29 (*)Find and sketch the running integral of:

- $x_1(t) = \delta(t) - \delta(t-2)$
- $x_2(t) = -\delta(t+3) + \delta(t-1) + 3\delta(t-3)$

30 (*)Find and sketch the derivative of:

- $x_1(t) = u(t+3) - 2u(t+3) + u(t+6)$
- $x_2(t) = 3u(t) - 2.5u(t-3)$
- $x_3(t) = u(t+1) + e^t u(t-3) - 2u(t)$
- $x_4(t) = \sin(\pi t)u(-t)$

31 Find the analytical expression and sketch the derivative of $x(t)$. Decompose $x(t)$ as a sum of unit steps.



Basic Signals

More Basic Signals

- **Rectangular Pulse** :

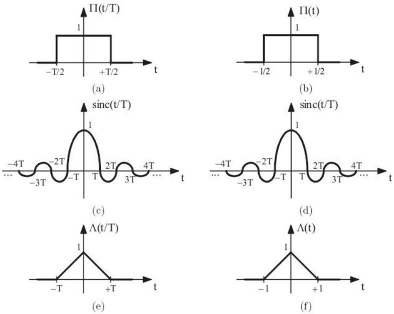
$$\Pi\left(\frac{t}{T}\right) = \begin{cases} 1; & -\frac{T}{2} \leq t < +\frac{T}{2} \\ 0; & otherwise \end{cases}$$

- **Sinc** (cardinal sine):

$$sinc\left(\frac{t}{T}\right) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}}$$

- **Triangular Pulse**:

$$\Lambda\left(\frac{t}{T}\right) = \begin{cases} \frac{t}{T} + 1; & -T \leq t \leq 0 \\ -\frac{t}{T} + 1; & 0 \leq t \leq +T \\ 0; & resto \end{cases}$$



Basic Signals

Questions

23 (*)Express $x(t) = \Pi(t)$ as a sum of shifted and scaled unit steps.

24 (*)Express $x(t) = u(t)$ as a sum of shifted and scaled rectangular pulses.

25 Sketch the rectangular pulse, the sinc and the triangular pulse for $T = 1$ and $T = 5$.

26 Sketch the following signals.

- $x_1(t) = \sum_{k=-\infty}^{\infty} (2\Lambda(t-5k) - \Lambda(t-2-5k))$
- $x_2(t) = \sum_{k=-\infty}^{\infty} (\Pi(t-5k) - \Pi(t-1-5k))$
- $x_3(t) = sinc(t-5\pi)$

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Reference

Reference for this topic

1. Chapter 1. *Signal and Systems*. A. V. Oppenheim, A.S. Willsky. Pearson Educación. 1997, 2^a edición.
2. Sections 1.0, 1.1, 1.2, 1.3 y 1.4.

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Problems

Problem 1 (*)

Express the following signals as complex exponentials:

1. $x(t) = 2\cos\left(2\pi 60t + \frac{\pi}{4}\right)$.
2. $x(t) = 2\cos\left(t + \frac{\pi}{6}\right) + 4\,\text{sen}\left(t - \frac{\pi}{3}\right)$.

[Sol: (a) $x(t) = e^{j2\pi 60t}e^{j\frac{\pi}{4}} + e^{-j2\pi 60t}e^{-j\frac{\pi}{4}}$. (b) $x(t) = -e^{j\left(t+\frac{\pi}{6}\right)} - e^{-j\left(t+\frac{\pi}{6}\right)}$]

Problem 2 (*)

Fing the magnitude and phase (as a function of t), as well as the average power and energy, for the following signals:

1. $x(t) = e^{j\left(2t+\frac{\pi}{4}\right)}$.
2. $x(t) = \cos(t)$.
3. $x(t) = e^{-2t}u(t)$.

[Sol: (a) $P_\infty = 1, E_\infty = \infty$. (b) $P_\infty = 1/2, E_\infty = \infty$. (c) $P_\infty = 0, E_\infty = 1/4$.]

Problems

Problem 3 (*)

Let be $x(t)$ a signal with $x(t) = 0$ para $t < 3$. For each of the following signals, find the values of t that makes $x(t) = 0$.

1. $x(1-t)$.
2. $x(t/3)$.
3. $x(3t)$.
4. $x(1-t) + x(2-t)$.
5. $x(1-t) \cdot x(2-t)$.

[Sol: (a) $t > -2$. (b) $t < 9$. (c) $t < 1$. (d) $t > -1$. (e) $t > -2$.]

Problem 4

Find the real part of the following signals and express them in the form $Ae^{-\alpha t} \cos \omega t + \phi$, where A, α, ω, ϕ are real numbers, with $A > 0$ and $-\pi \leq \phi \leq \pi$.

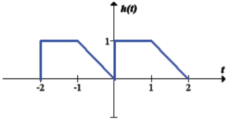
1. $x(t) = -2$.
2. $x(t) = \sqrt{2} e^{j\pi/4} \cos(3t + 2\pi)$.
3. $x(t) = e^{-t} \sin(3t + \pi)$.
4. $x(t) = je^{(-2+j100)t}$.

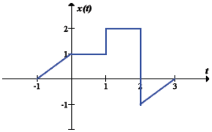
[Sol: (a) $A = 2, \alpha = 0, \omega = 0, \phi = \pi$. (b) $A = 1, \alpha = 0, \omega = 3, \phi = 0$.
(c) $A = 1, \alpha = 1, \omega = 3, \phi = \pi/2$. (d) $A = 1, \alpha = 2, \omega = 100, \phi = \pi/2$.]

Problems

Problem 5 (*)

Given the signals $x(t)$ and $h(t)$, sketch each of the following signals.





1. $h(t+3)$.
2. $h\left(\frac{t}{2}-2\right)$.
3. $h(1-2t)$.
4. $4h\left(\frac{t}{4}\right)$.
5. $h\left(\frac{t}{2}\right)\delta(t+1)$.
6. $h(t)[u(t+1)-u(t-1)]$.
7. $x(t)h(t+1)$.
8. $x(t)h(-t)$.

Problems

Problem 6

Determine whether the following signals are periodic. If they are, find the period.

1. $x(t) = 2\cos\left(3t + \frac{\pi}{4}\right)$.
2. $x(t) = e^{j(\pi t - 1)}$.
3. $x(t) = 2\cos\left(\frac{\pi}{4}t\right) + \sin\left(\frac{\pi}{8}t\right) - 2\cos\left(\frac{\pi}{2}t + \frac{\pi}{6}\right)$.

[Sol: (a) $T = 2\pi/3$s. (b) $T = 2$s. (c) $T = 16$s.]

Problem 7 (*)

Find the derivative of the following signals:

1. $x(t) = \begin{cases} 0, & t < 1 \\ 2, & 1 \leq t < 2 \\ -1, & 2 \leq t < 4 \\ 1, & t \geq 4 \end{cases}$.
2. $x(t) = u(t+2) - u(t-2)$.
3. $x(t) = e^{j\pi t}u(t)$.

Problems

Problem 8 (*)

Integrate the following signals computing $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$:

1. $x(t) = \delta(t+2) - \delta(t-2)$.
2. $x(t) = u(t+2) - u(t-2)$.
3. $x(t) = e^{j\pi t}u(t)$.

[Sol: (a) $y(t) = u(t+2) - u(t-2)$. (b) $y(t) = (t+2)u(t+2) + (2-t)u(t-2)$. (c) $y(t) = -\frac{j}{\pi}\left(e^{j\pi t} - 1\right)u(t)$.]

Problem 9

Let be $x(t) = \delta(t+2) - \delta(t-2)$. Determine the total energy of $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$.
[Sol: $E_\infty = 4$ J.]

Problems

Problem 10 (*)

Let be a periodic signal with period $T = 2$, given by:

$$x(t) = \begin{cases} 1, & 0 \leq t < 1 \\ -2, & 1 \leq t < 2 \end{cases}$$

The derivative of this signal is related with the impulse train with period 2 sec, given by:

$$g(t) = \sum_{k=-\infty}^{\infty} \delta(t - 2k)$$

Determine the values A_1, t_1, A_2, y t_2, so that

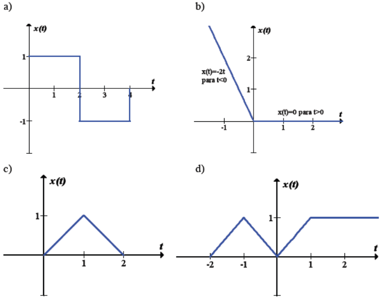
$$\frac{dx(t)}{dt} = A_1 g(t - t_1) + A_2 g(t - t_2)$$

[Sol: $A_1 = 3, t_1 = 0, A_2 = -3, t_2 = 1.$]

Problems

Problema 11

Sketch the even and ood part of the following signals:



Problem 12 (*)

Show that $\delta(2t) = \frac{1}{2}\delta(t)$.

Problems

Problem 13 (*)

We define the function $\Phi_{xy}(t)$ of two signals $x(t)$ and $y(t)$ as:

$$\Phi_{xy}(t) = \int_{-\infty}^{\infty} x(t+\tau)y(\tau)d\tau$$

- What is the relationship between $\Phi_{xy}(t)$ and $\Phi_{yx}(t)$?
- Let's suppose that $x(t)$ is periodic. Is also periodic $\Phi_{xx}(t)$? If so, what is the period?
- Find the odd part of $\Phi_{xx}(t)$.

Problem 14 (*)

Show that $\int_{-\infty}^{\infty} x^2(t)dt = \int_{-\infty}^{\infty} x_{par}^2(t)dt + \int_{-\infty}^{\infty} x_{impar}^2(t)dt$.