Topic 1: Review of Stochastic Processes

Telecommunication Systems Fundamentals

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Academic year 2.013-2.014

Universidad Rey Juan Carlos

Concepts in this Chapter

- Review of Signals models and classification
  - Examples of actual signals
  - Signal modeling
  - Signals classification
- Review of Statistical Basics: Modeling of Stochastic Processes
  - Amplitude distribution (probability density function, pdf) and averages
  - Autocorrelation
  - Independence
  - Stationarity
  - Ergodicity
  - Cross-correlation
  - Power and Energy Spectral Density

Theory classes: 3 sessions (6 hours)
Problems resolution: 1 session (2 hours)
Lab (Matlab): 2 hours

Bibliography

1. Communication Systems Engineering. John. G. Proakis. Prentice Hall

2. Sistemas de Comunicación. S. Haykin. Wiley

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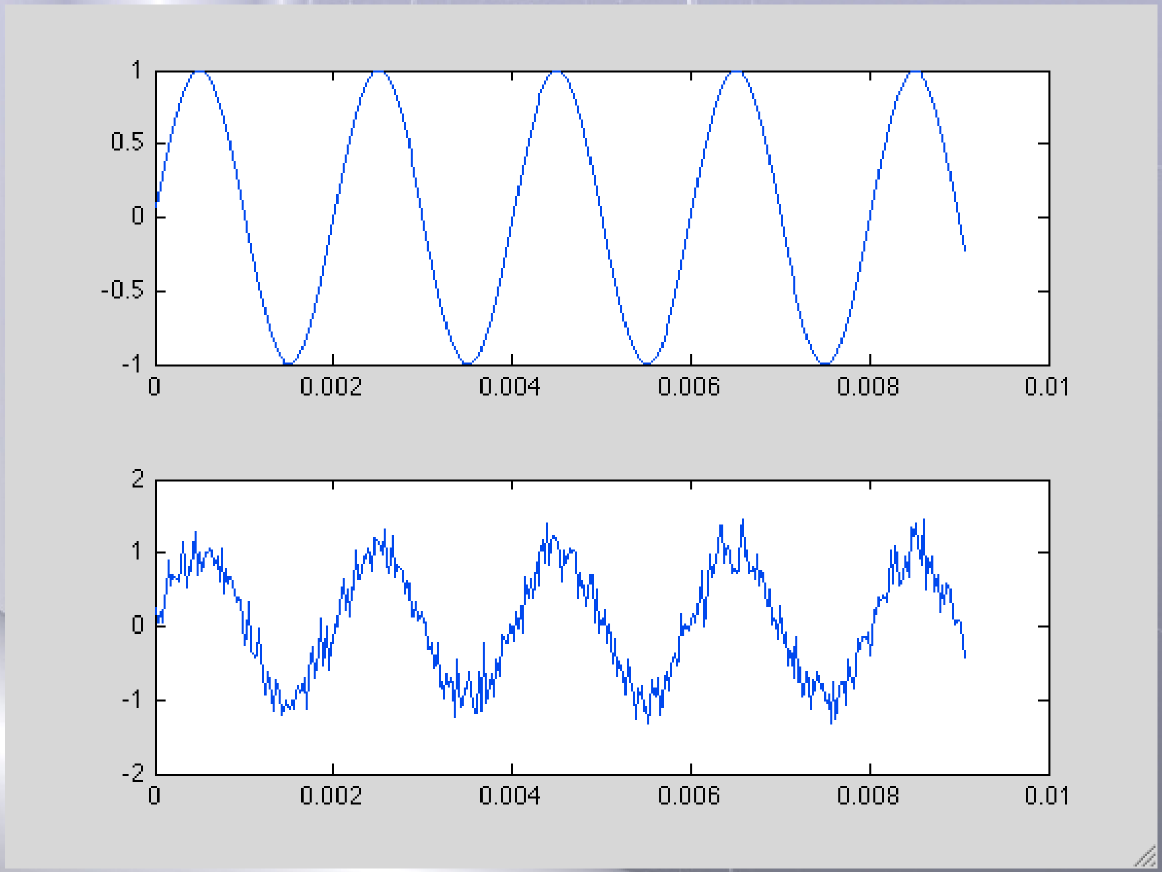
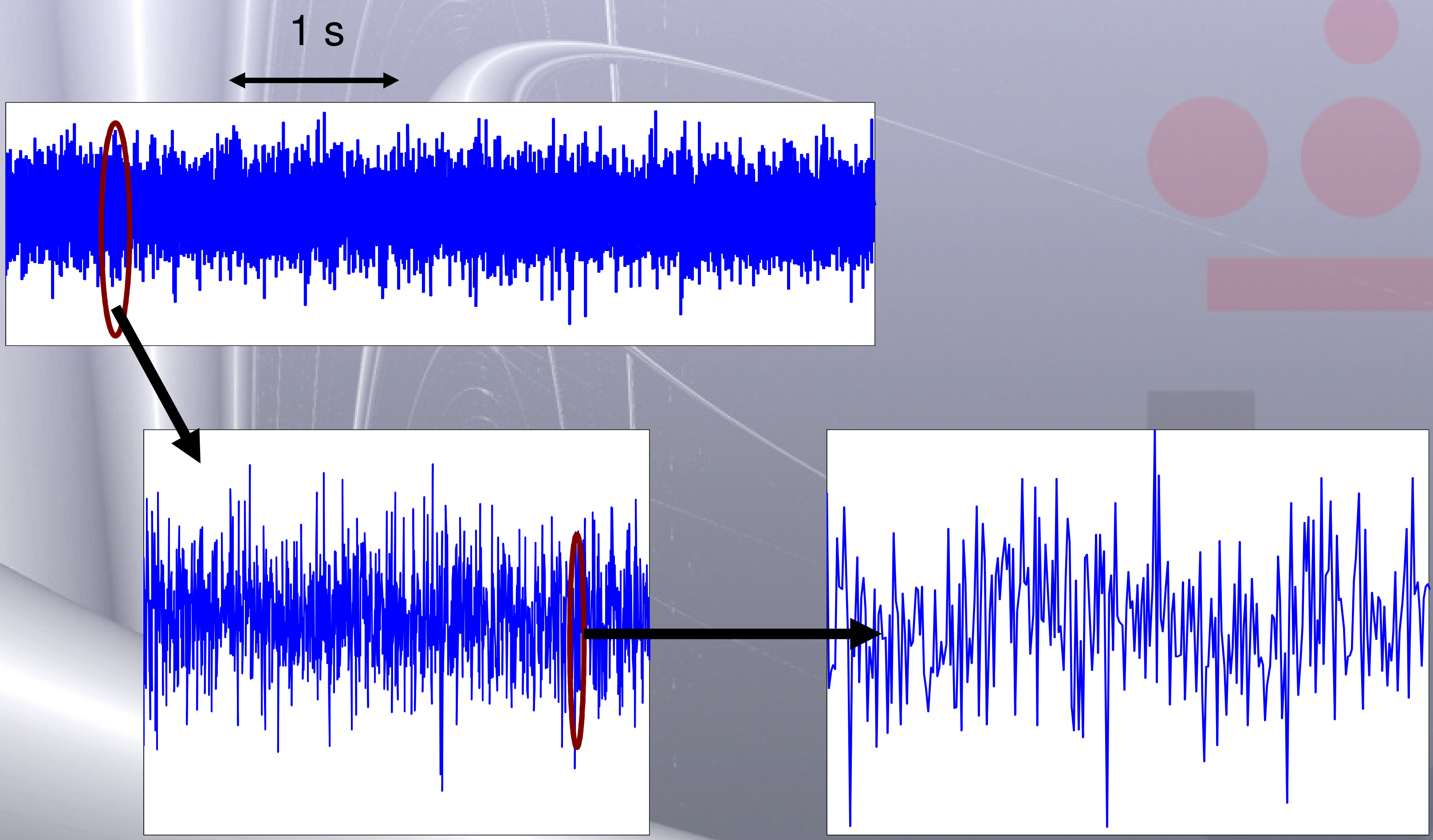
Pure Tone



Noise Free

Aditive Noise

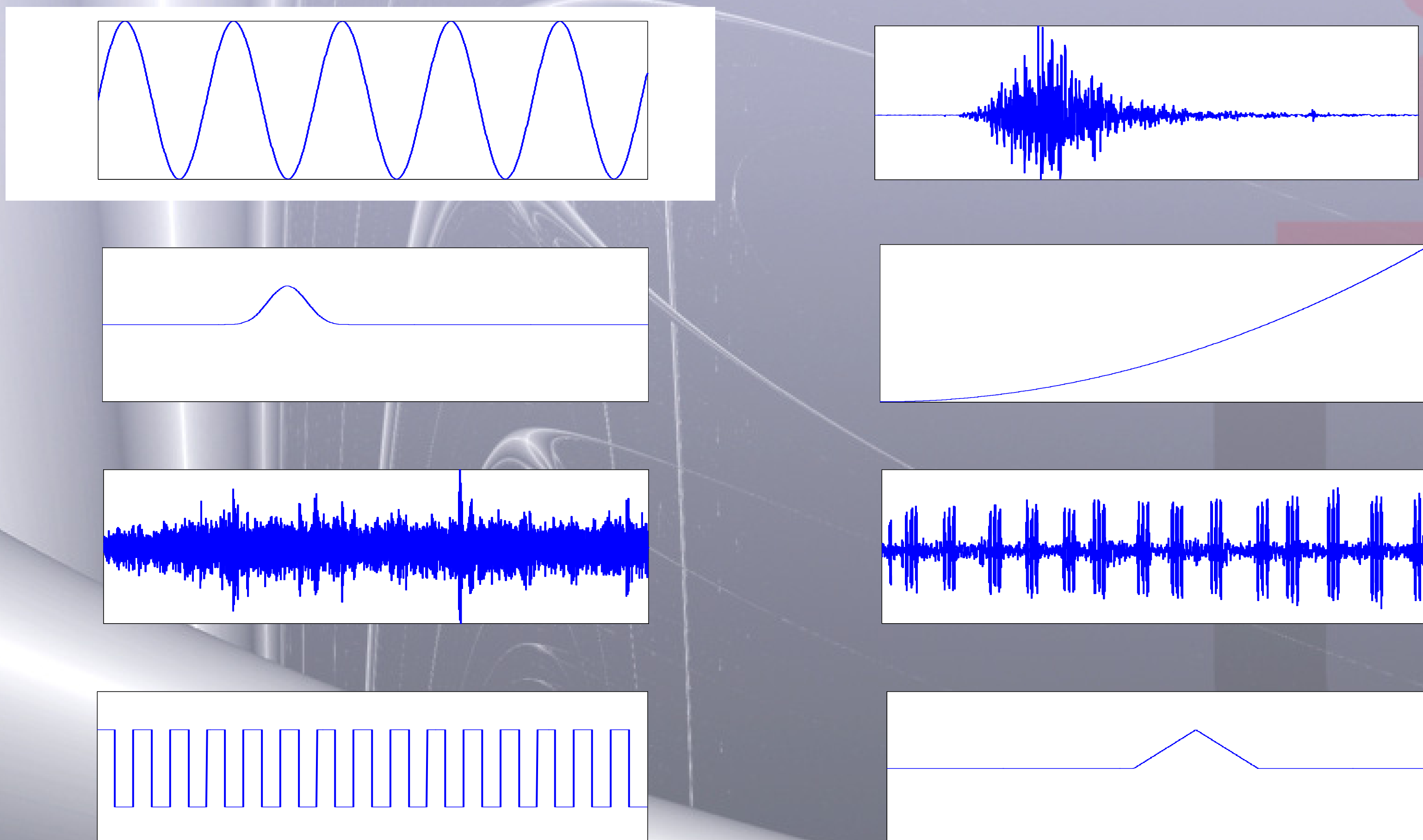
All frequencies



Why modeling Signals?

- To answer the following questions:
  - What information does the signal contain? How is the info coded into the signal? How much info does the signal contain?
  - How does the channel affect the transmitted signal?
  - How is the telecommunication system designed?
- We describe signals by their mathematical model – measurable characteristics of the signal

How would you describe them?



Signal Modeling

- In a point-by-point description, the value of the signal at each time instant is stored in a look-up-table

t	$x(t)$
…	…
0	7
1	2
…	…

- The point-by-point description is valid for any signal (assuming the sampling rate is fast enough) and contains all the information within the signal, but “seeing” the information is not evident

Signal Modeling

- Some signals can be modeled by a mathematical expression that provides its amplitude as function of time

$$x(t) = \sum_{n=0}^{\infty} A_n \cos(n\omega_0 t)$$

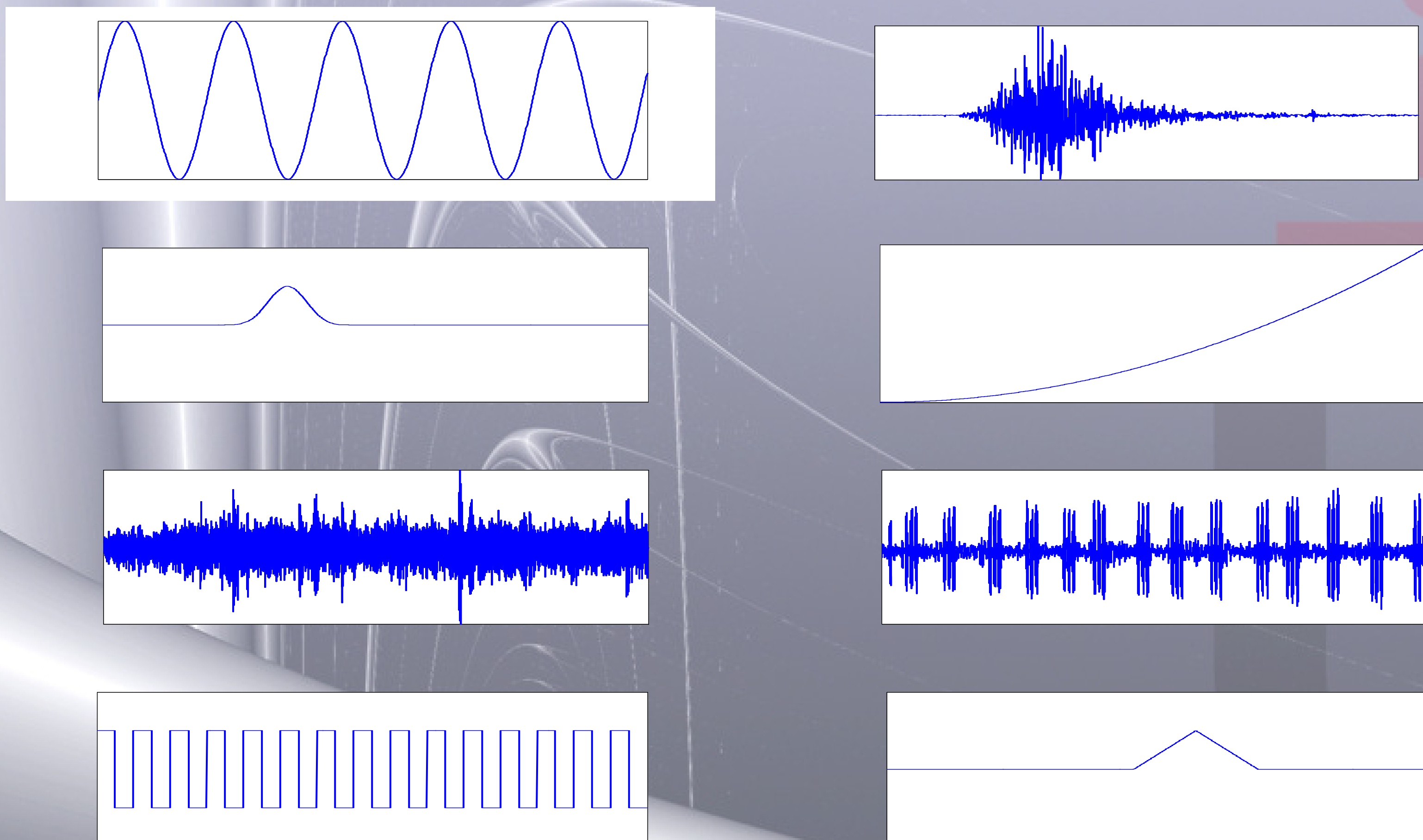
- This type of signals are named “Deterministic” because their lack of randomness
  - Only few signals in telecommunications systems can be modeled as simple as this

Signal Modeling

- We can briefly describe a signal by some of its characteristics
  - Mean value
  - Mean squared value (power)
  - Energy
  - Standard deviation
  - Autocorrelation
  - …

- It is a universal procedure (usable for any kind of signal), and it gives some criteria to classify signals. However, it does not describe the signal completely (univocally).

How would you describe them?



Mean Value

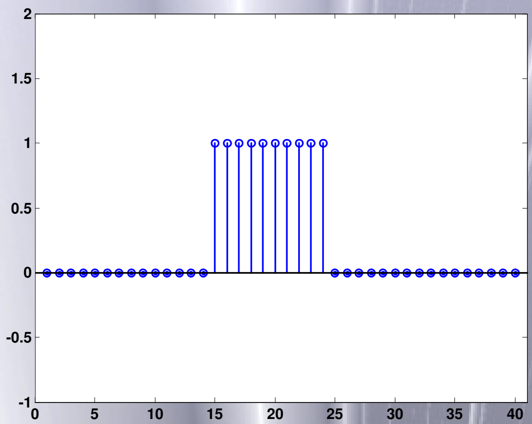
- For time-discrete signals, mean value is defined as:

$$\langle x[n] \rangle = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]$$

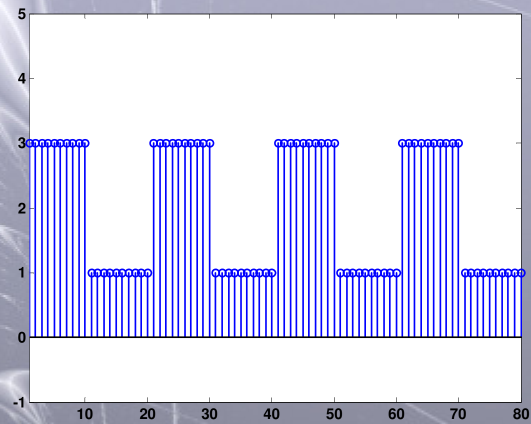
- For time-continuous signals, mean value is defined as:

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)dt$$

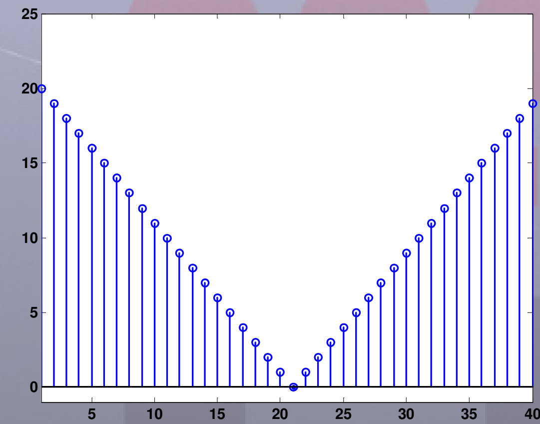
Mean Value



$$\langle x_1(t) \rangle = 0$$



$$\langle x_2(t) \rangle = 2$$



$$\langle x_3(t) \rangle = \infty$$

Energy

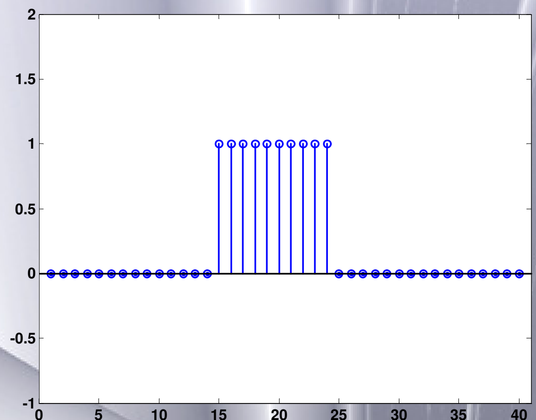
- A decisive classification of signals is related to its energy and power: finite energy, or power defined. For finite energy signals, it is defined
  - For discrete signals:

$$\mathbf{E_x} = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

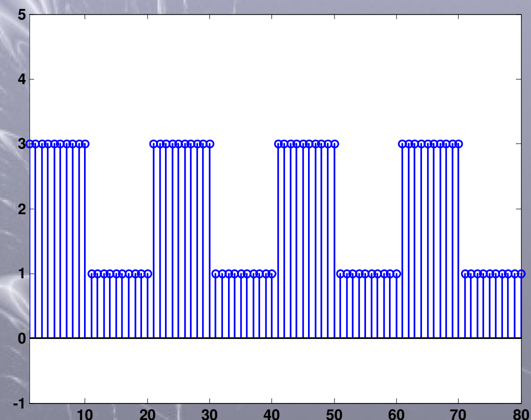
  - For continuous signals:

$$\mathbf{E_x} = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt$$

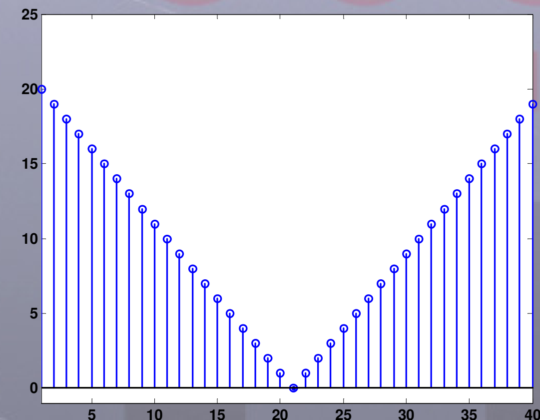
Energy



$$E_{x_1} = 10$$



$$E_{x_2} = \infty$$



$$E_{x_3} = \infty$$

Average Power

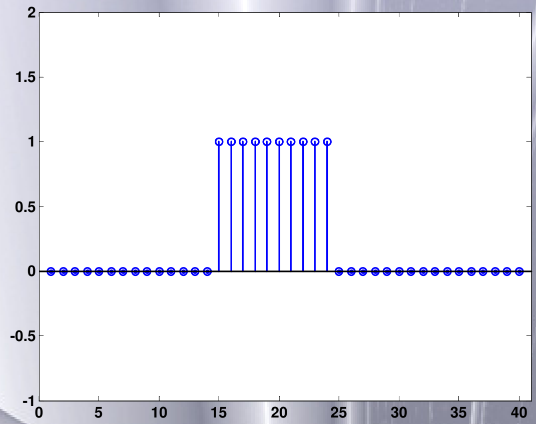
- The average power of discrete signals is defined as:

$$\mathbf{P_x} = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

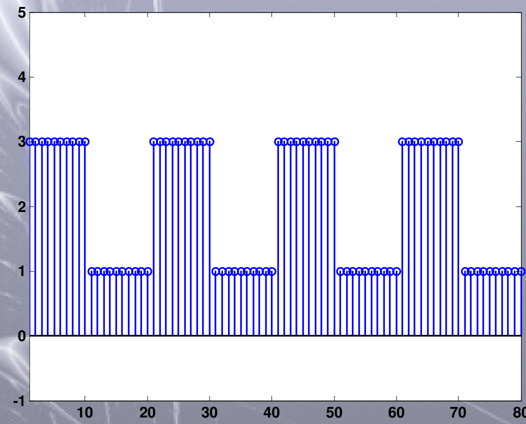
- While the average power of continuous signal is defined as:

$$\mathbf{P_x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$

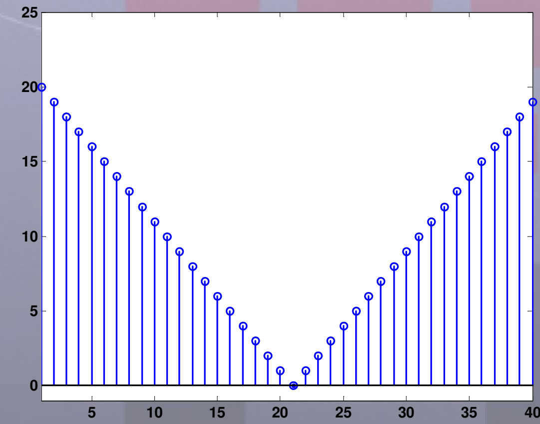
Average Power



$P_{x_1} = 0$



$P_{x_2} = 5$



$P_{x_3} = \infty$

Goes to zero when the length of the samples increases

A signal classification

Signal

- Finite Energy (Energy Defined)

$$0 \le \mathbf{E_x} < \infty$$
$$\mathbf{P_x} = 0$$

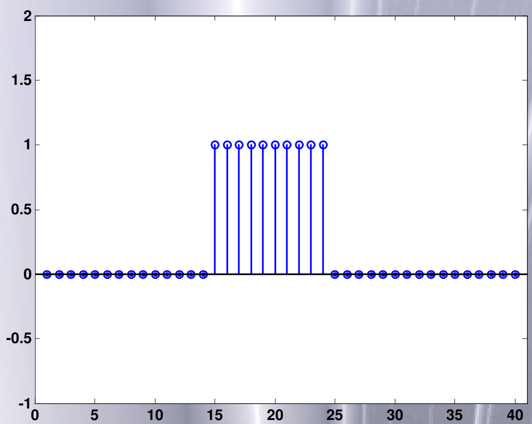
- Finite Average Power (Power Defined)

$$\mathbf{E_x} = \infty$$
$$0 < \mathbf{P_x} < \infty$$

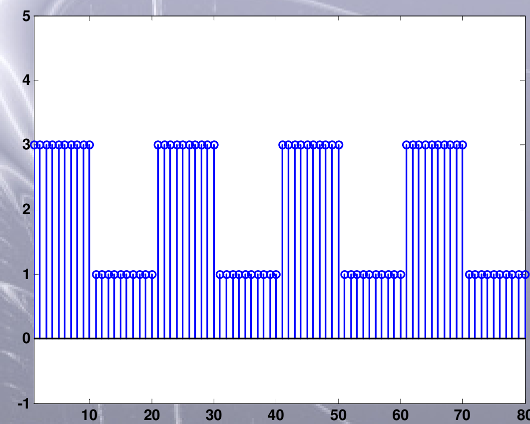
- Unbounded Average Power

$$\mathbf{E_x} = \infty$$
$$\mathbf{P_x} = \infty$$

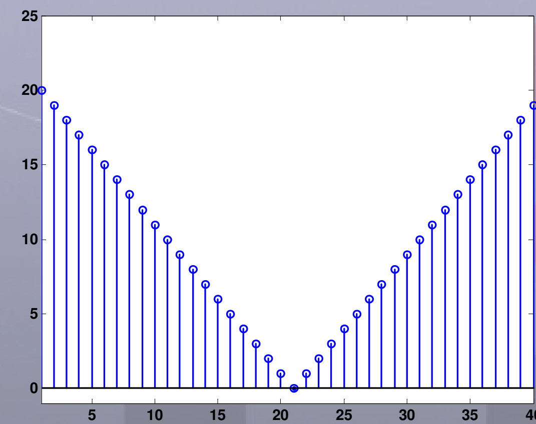
Energy/Power signal classification



Finite
Energy
(Energy Defined)



Finite
Average
Power
(Power Defined)



Unbounded
Average
Power
(assuming it
increases
indefinitely)

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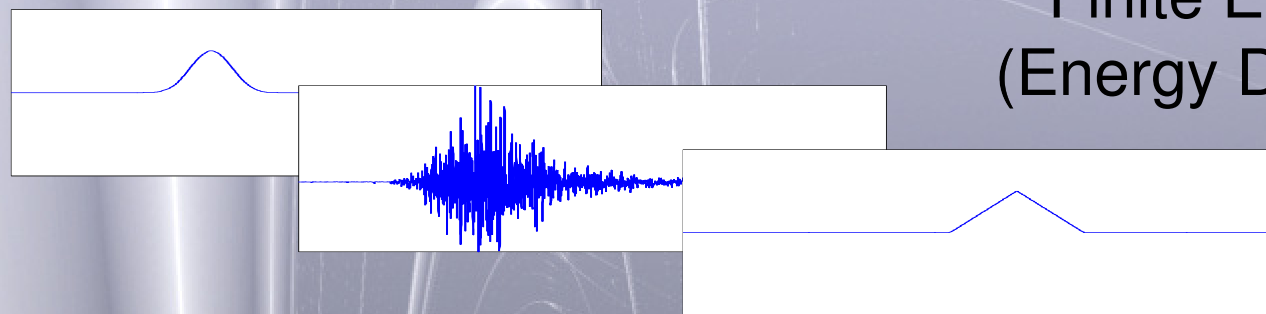
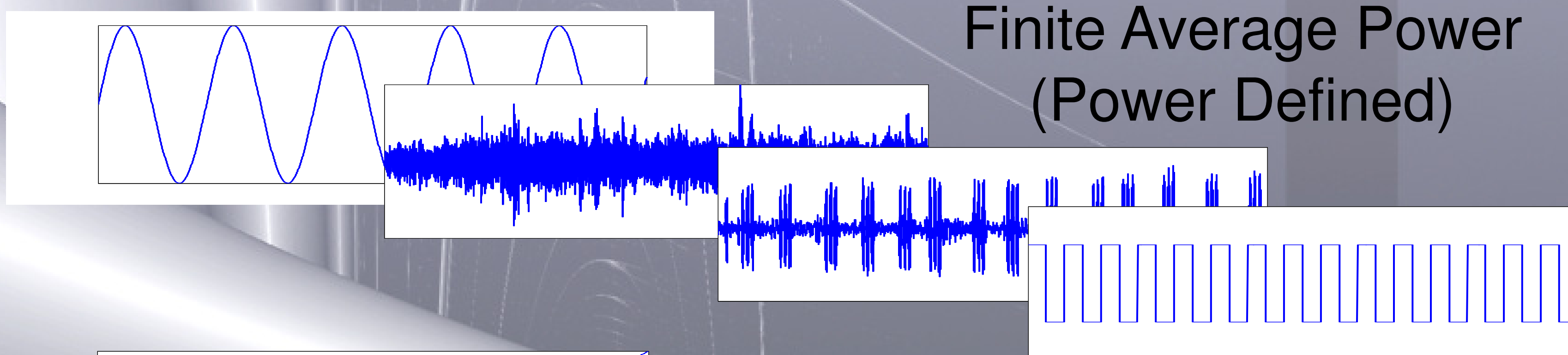
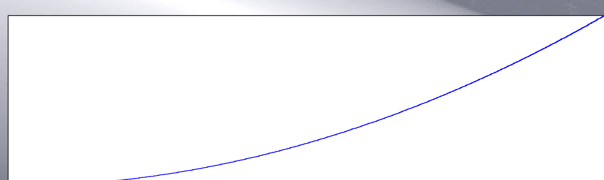
Energy/Power signal classification



Finite Energy
(Energy Defined)



Finite Average Power
(Power Defined)



Unbounded Average Power

Homework

- Compute: i) Average Value; ii) Energy and ii) Average Power of the two following signals:

$$x_1(t) = A\cos(2\pi f t)$$

$$x_2(t) = Ae^{j2\pi f t}$$

Classifying Signals: A Taxonomy

- Continuous / Discrete
  - Analog / Digital
- Deterministic / Stochastic (random signals)
  - Deterministic:
    - Energy Defined (time limited)
    - Power Defined
      - Periodic / Non periodic
  - Stochastic
    - Stationary
      - Ergodic / Non-Ergodic
    - Non-Stationary
- Other classifications
  - Real valued / Complex
  - Even / Odd
  - Hermitical / Non-Hermitical

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Recall: Time Averaging and Expected Value

$$\langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$\updownarrow$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

Homework: Time Averaging and Expected Value

- Generate a Random Variable uniformly distributed between 0 and 1
  - X ~U(0,1)
  - Pdf - f(x) = 1, for 0 < x < 1, and 0 otherwise.

1. Run a simulation (Matlab) of 10.000 samples of U(0,1)
2. Compute the average value of the 10.000 samples
3. Analytically calculate expected value of U(0,1)
4. Compare values obtained in 2 and 3.

Recall: Time Averaging and Expected Value

$$\left\langle x^2(t) \right\rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$$\Updownarrow$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

Homework: Time Averaging and Expected Value

- Generate a Random Variable uniformly distributed between 0 and 1
  - X ~U(0,1)
  - Pdf - f(x) = 1, for 0 < x < 1, and 0 otherwise.

1. Run a simulation (Matlab) of 10.000 samples of U(0,1)
2. Compute the average power of the 10.000 samples
3. Analytically calculate second moment of U(0,1)
4. Compare values obtained in 2 and 3.

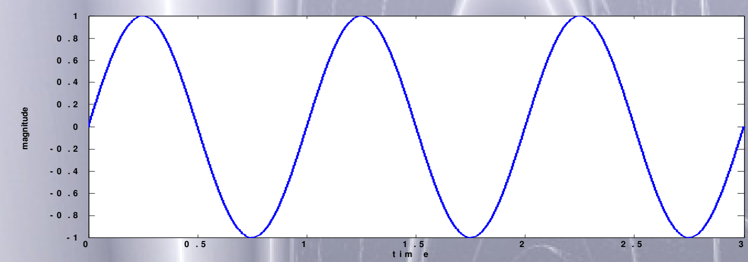
Why statistical modeling is useful?

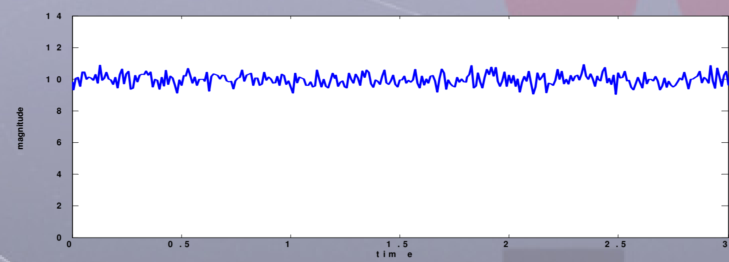
1. Characterizing a stochastic process would require the specification of the signal at every single instant
2. Most cases we do not know the signal ***a priori***
3. We get the whole signal in very rare occasions

Statistical model to:

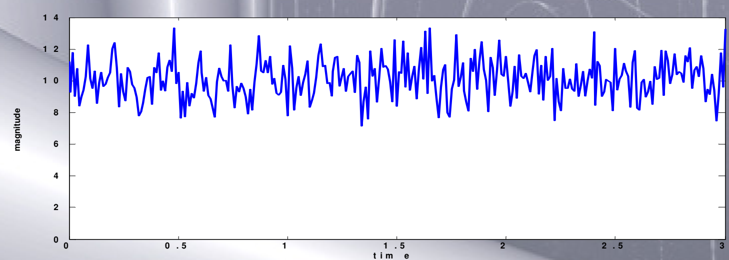
- … **sumarize** the description of a signal behaviour
- … describe **sets of** signals
- … describe the whole signal from a **finite time interval**

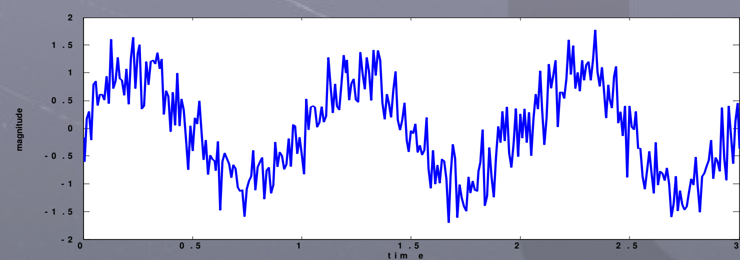
How do you describe them?



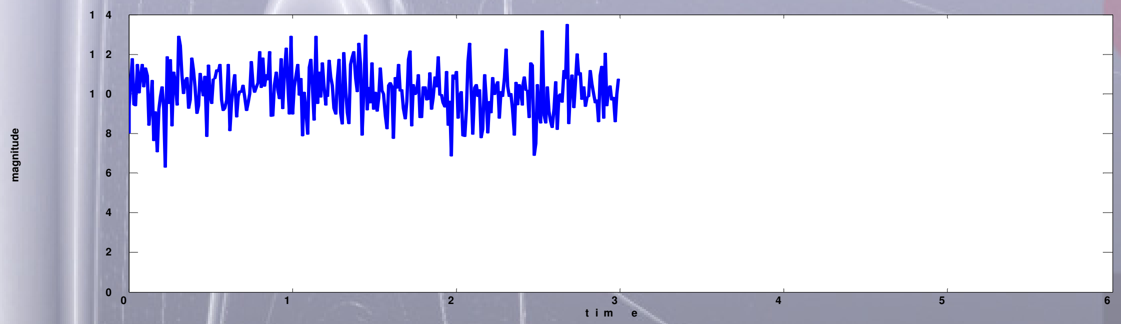


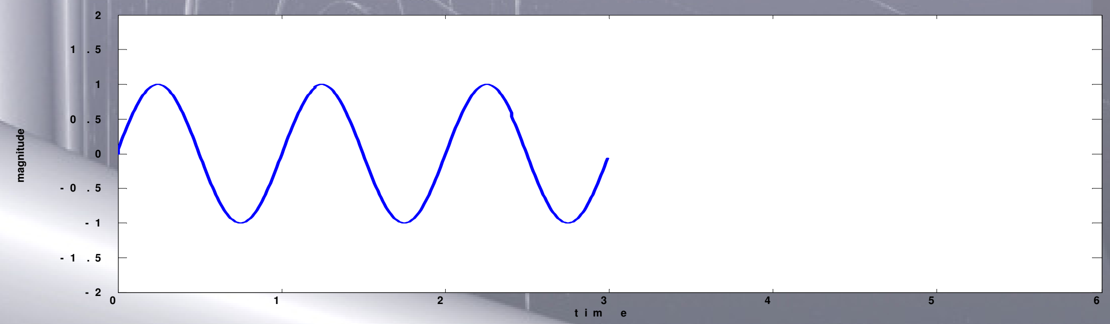
Why Statistical Modeling is Useful?





What’s next?





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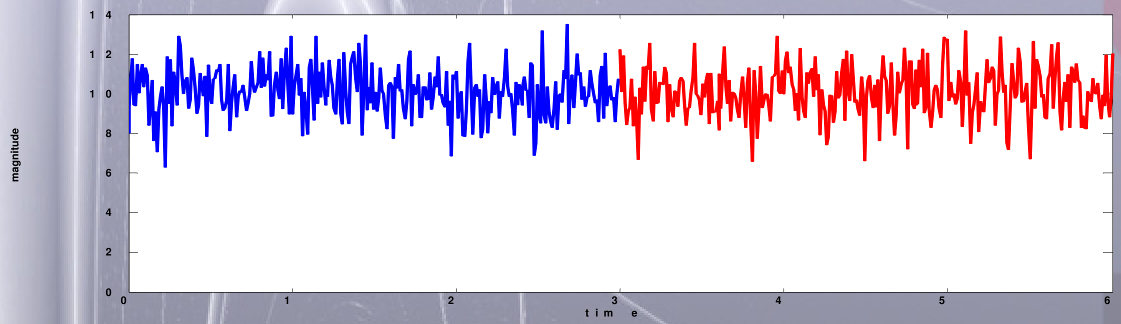
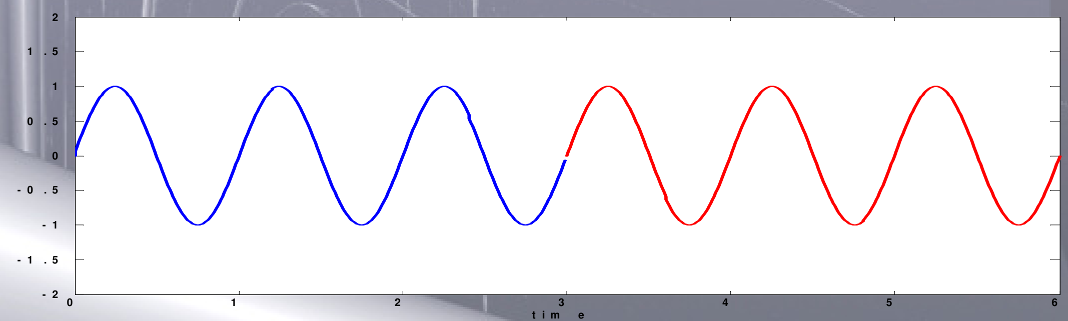
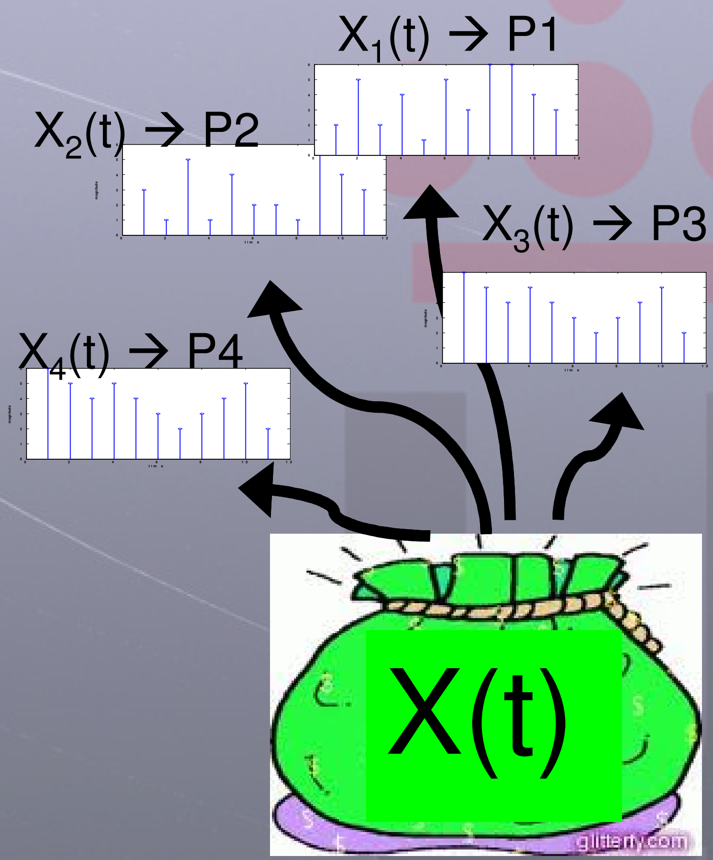
What's next?





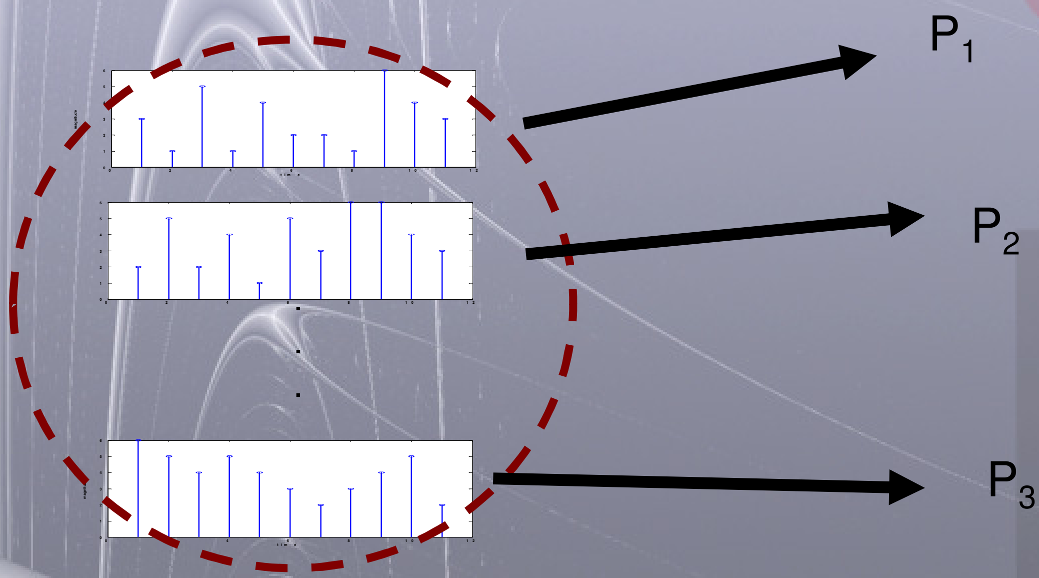
Review of the Concept of Stochastic Process

- Definition 1: a SP can be seen as series of Random Variables; or it can be also seen as a RV that is time-variant
- Definition 2: a SP can be seen as a set of time-variant signals, each one with its probability of happening (imagine a bag with all the possible signals and you get one of them).
- Note: we talk about "time" when referring to the independent variable, but it could be different:
  - *Example: thickness of bar as function of its length*



Stochastic Process Model

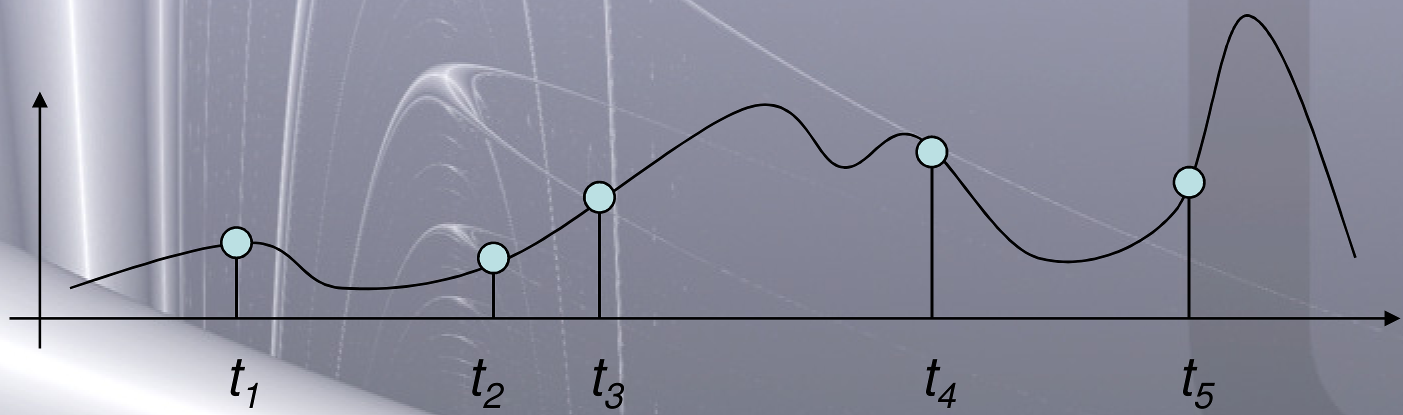
- To fully characterize an SP a probability measurement of each possible realization has to be provided.



- In other words, we should be able to tell how likely is any given observation (realization) to happen

Stochastic Process Model

- A complete description of a SP, $X(t)$, requires the definition of the sequence $(X(t_1), \ldots, X(t_k))$ for any value of k and any value of the k-tuple $(t_1, \ldots, t_n)$.



Stochastic Process Model

- In general practice, we will not look for a complete description of the SP, but we will define by two main aspects:
  - Amplitude distribution
  - Autocorrelation, which contains the time variation description (statistical relationship betwen two instants of the signal)

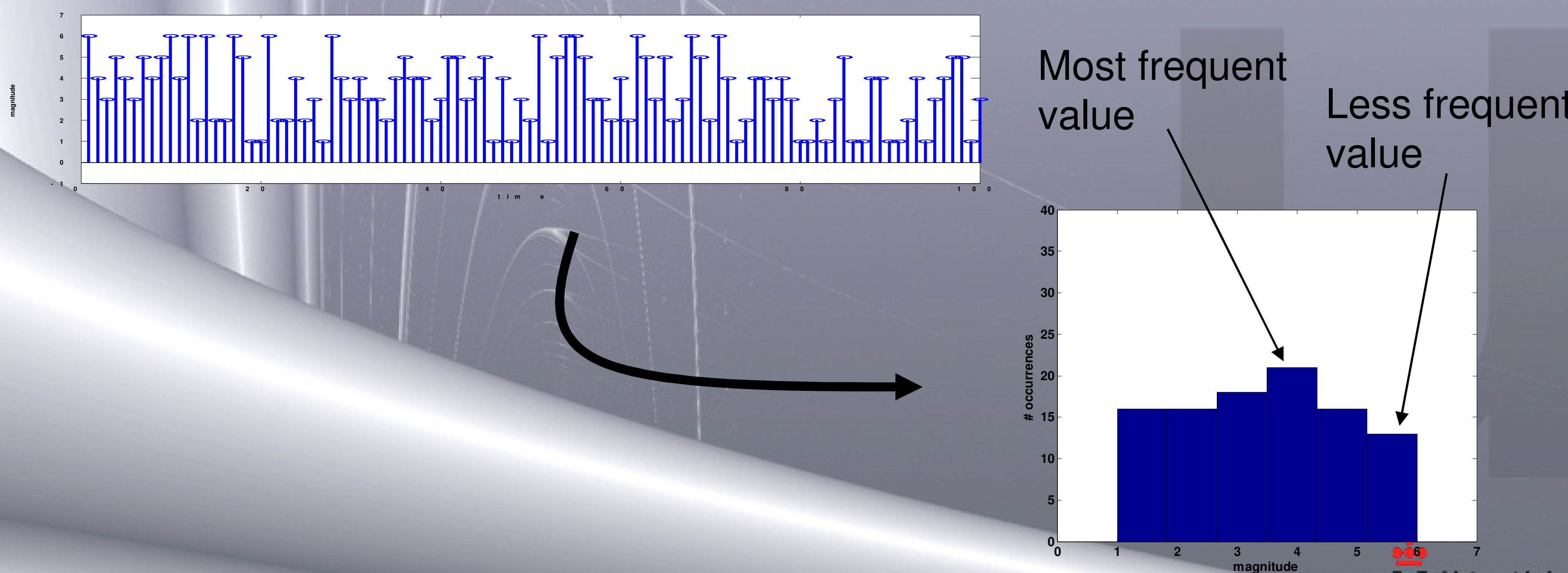
- Autocorrelation can be expressed also as Power Spectral Density – the Fourier Transform of the Autocorrelation
- Later, we can analyse the impact of a linear channel on the SP, i.e. the impact of the channel on the amplitude distribution and on the autocorrelation

Summarizing main concepts of SP

- A SP is a mechanism that generates time-variant amplitudes – a signal. Each of the signal produced by a SP is called “realization”
- The SP model also applies to each realization. In other words, a model for a SP models also every possible realization of it.
- We will model SP by their amplitude distribution and autocorrelation. Amplitude distribution models the realization values at a given time, and autocorrelation models the time variation of the SP.
- By computing Fourier Transform of autocorrelation we get the Power Spectral Density – the information of the amount of energy contained in each frequency – the spectrum
- Noise in telecommunications is modeled as a SP

Amplitude Distribution

- For each instant of time of the SP (sample), its amplitude is a Random Variable following a Probability Density Function (pdf)
- Amplitude Distribution is modeled by its pdf. It can be modeled both by its amplitude (two values for complex signals) or by its power. Thus, a pdf of the signal amplitude or a pdf of the signal power should be provided



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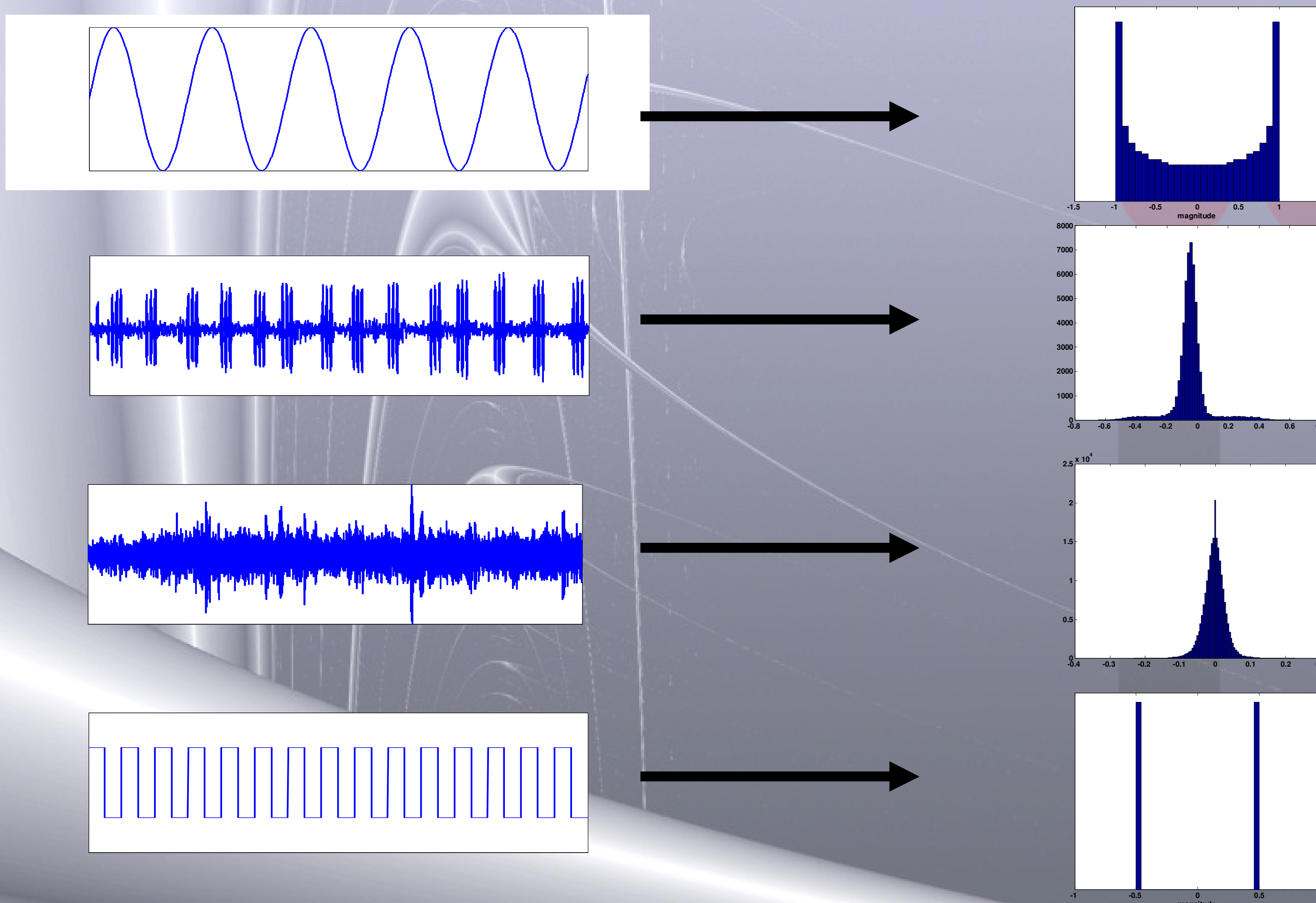
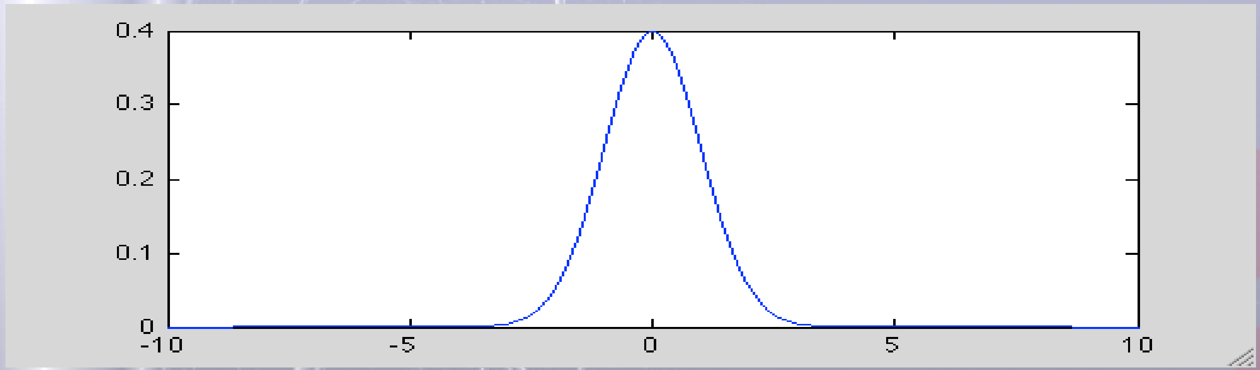
Amplitude Distribution



Homework

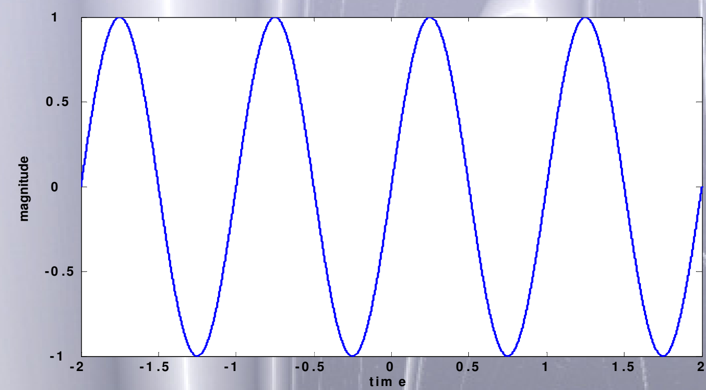


- Specify the magnitudes on each axes of the above graph
  - A) If we interpret the plot as a Gaussian-shaped signal
  - B) If we interpret the plot as the pdf of the voltage of a noisy signal

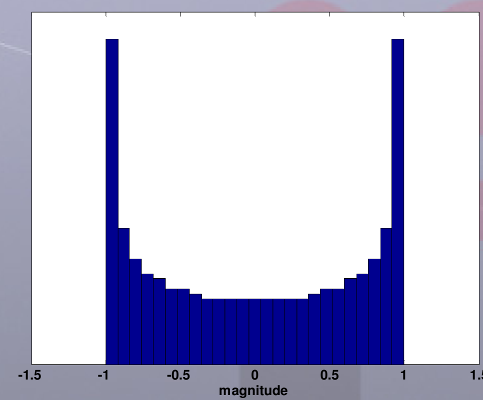
- What is the mean value for each case?

Mean value: time domain and statistical approach

- Time average



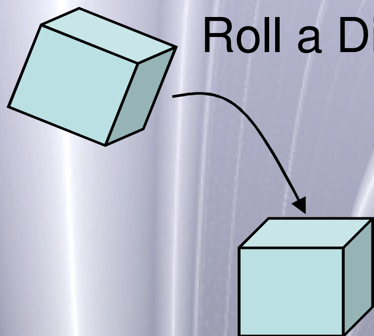
- Statistical mean value



$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt \quad \leftrightarrow \quad E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt \quad \leftrightarrow \quad E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

Numerical Example



- Time average

x[n] = 5 6 2 1 2 3 2 6 1 4 1 6 2 4 6
3 1 2 3 3 2 6 5 6 3 6 1 4 5 2
6 5 5 4 5 2 2 2 4 5 2 3 6 4 6
2 5 5 2 3 3 4 5 1 5 6 3 3 4 5
5 5 3 3 6 4 6 2 4 4 6 1 4 4 1
6 ...

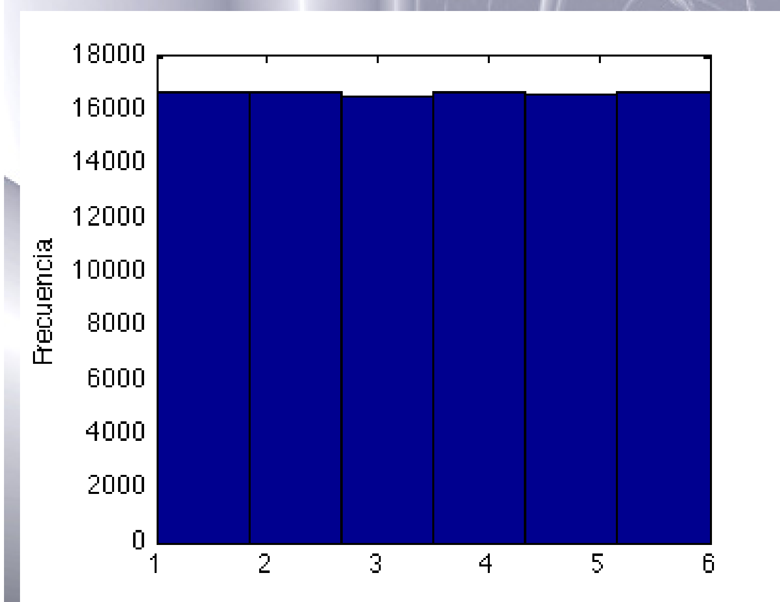
$$\rangle x[n] \langle = \frac{1}{N}\sum_{n=0}^{N-1} x[n] = 3{,}7$$

3,7 for 100 samples, for 100.000 → 3,4997

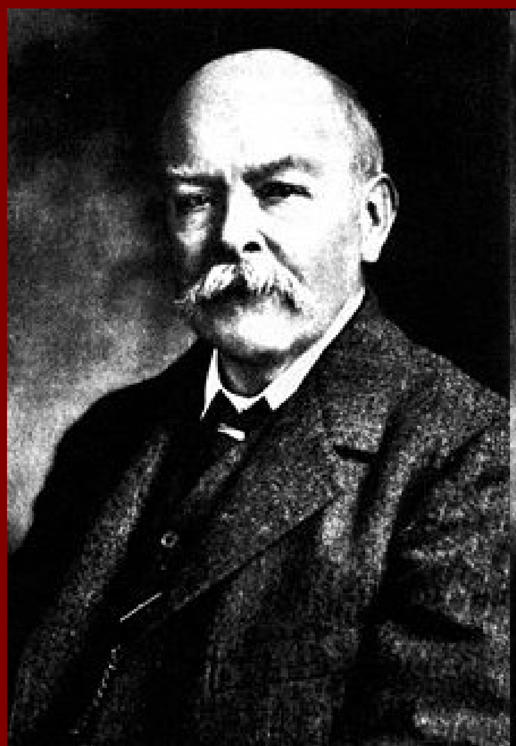
- Statiscal average

$$E(X) = \sum_{n=1}^{N} n f_x[n] =$$

$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = 3.5$$



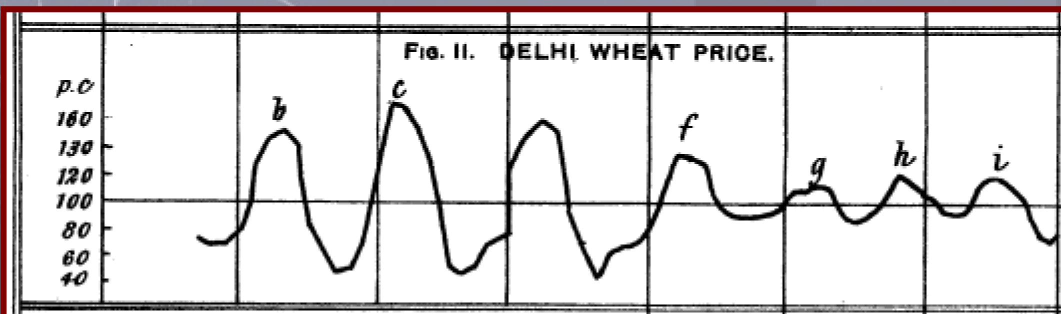
The Correlogram



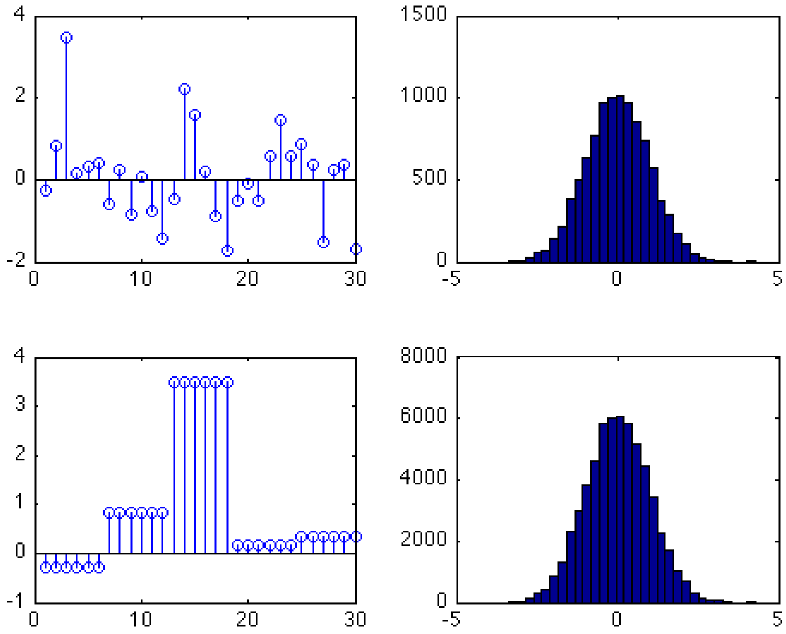
John Henry Poynting
(1852 - 1914)

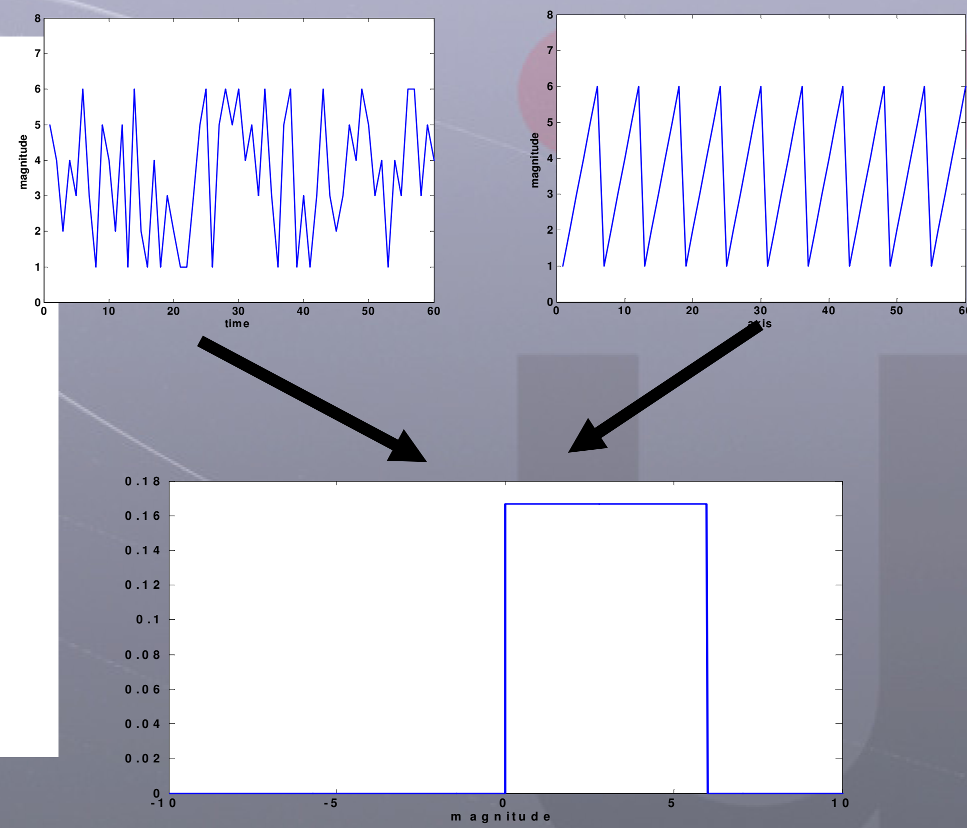
A COMPARISON *of the* FLUCTUATIONS *in the* PRICE *of* WHEAT *and in the* COTTON *and* SILK IMPORTS *into* GREAT BRITAIN. *By* J. H. POYNTING, M.A., *late Fellow of Trinity College, Cambridge; Professor of Physics, Mason College, Birmingham.*

[Read before the Statistical Society, 15th January, 1884. Sir RAWSON W. RAWSON, K.C.M.G., C.B., a Vice-President, in the Chair.]



Amplitude distribution is not enough to describe a time-variant SP





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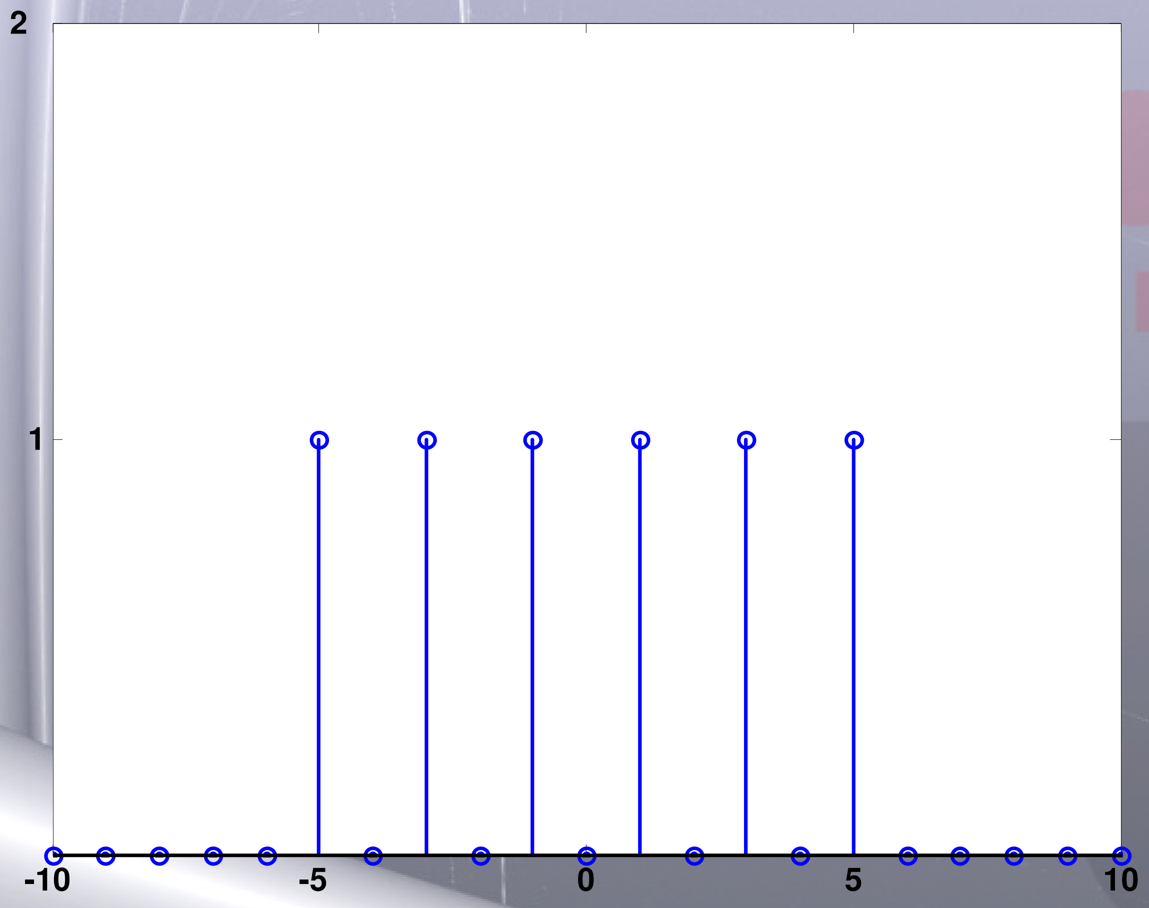
Autocorrelation

- Many power-defined signals (time-unbounded) exhibit repetition patterns. Although such signals are not periodic, they have some periodicity on their amplitude distribution. They are quasi-periodic
- How can we study such signals?
- An approach to analyze quasi-periodic patterns is to check the likeness between the signal and a delayed version of itself
- The autocorrelation function describes the likeness of a signal with a delayed version of itself. Therefore, we can identify quasi-periodic patterns by computing the signal autocorrelation

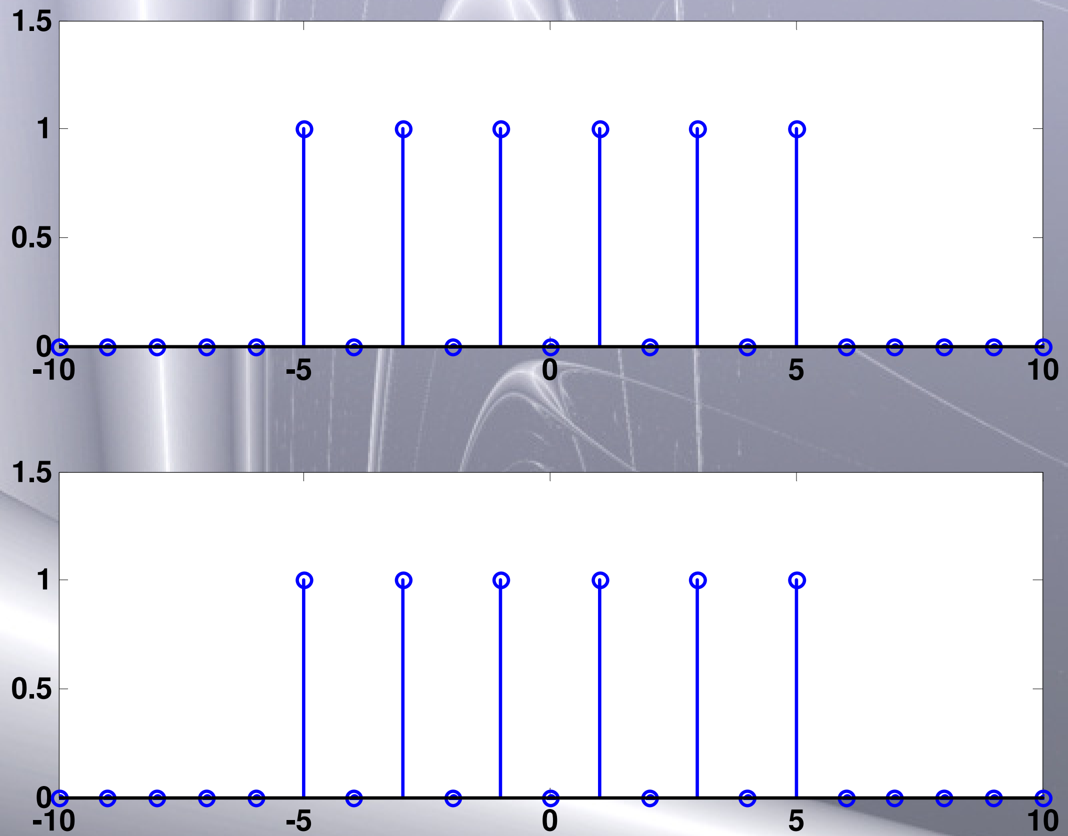
Autocorrelation

- To measure likeness between a given signal and a delayed version of itself we use the inner product of both signals. So, autocorrelation is defined that way.
- The likeness measurement (inner product) is defined in different way for power-defined and energy-defined signals
- Calculation of inner product depends on the available information of the signal
  - If time description of realizations is available, we can compute inner product as time average
  - If statistical information is available, inner product will be computed as statistical average

Example 1: Energy-Defined Signal



Example 1: Energy-Defined Signal



…

$x(-5)x(-5) = 1$

$x(-4)x(-4) = 0$

$x(-3)x(-3) = 1$

$x(-2)x(-2) = 0$

$x(-1)x(-1) = 1$

$x(0)x(0) = 0$

$x(1)x(1) = 1$

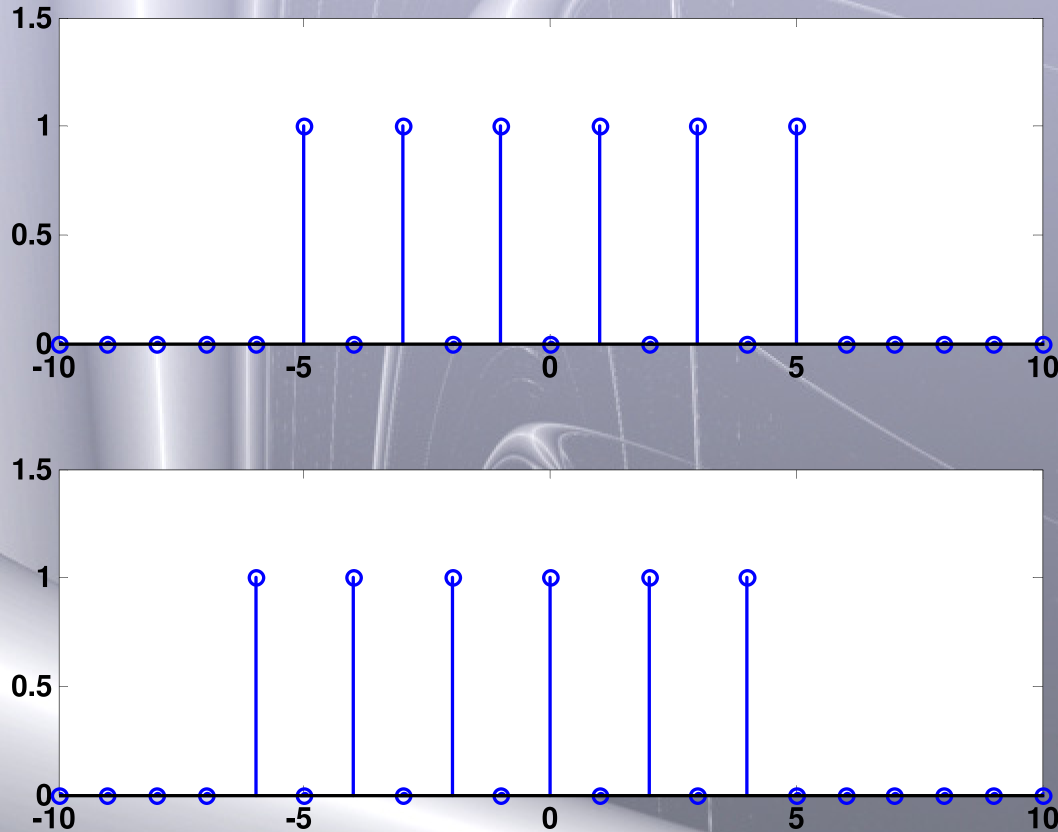
$x(2)x(2) = 0$

$x(3)x(3) = 1$

…

$\Sigma x(n)x(n) = 6$

Example 1: Energy-Defined Signal



...

$x(-5)x(-4) = 0$

$x(-4)x(-3) = 0$

$x(-3)x(-2) = 0$

$x(-2)x(-1) = 0$

$x(-1)x(0) = 0$

$x(0)x(1) = 0$

$x(1)x(2) = 0$

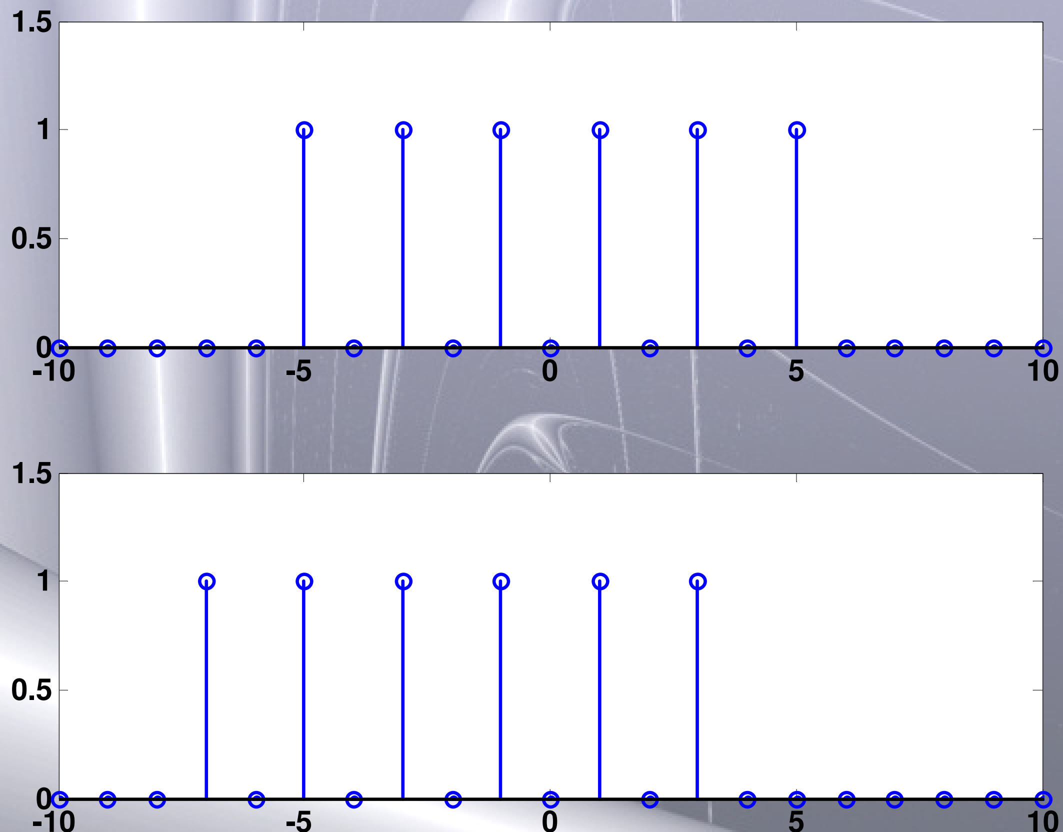
$x(2)x(3) = 0$

$x(3)x(4) = 0$

...

$\Sigma x(n)x(n+1) = 0$

Example 1: Energy-Defined Signal



…

$x(-5)x(-3) = 1$

$x(-4)x(-2) = 0$

$x(-3)x(-1) = 1$

$x(-2)x(0) = 0$

$x(-1)x(1) = 1$

$x(0)x(2) = 0$

$x(1)x(3) = 1$

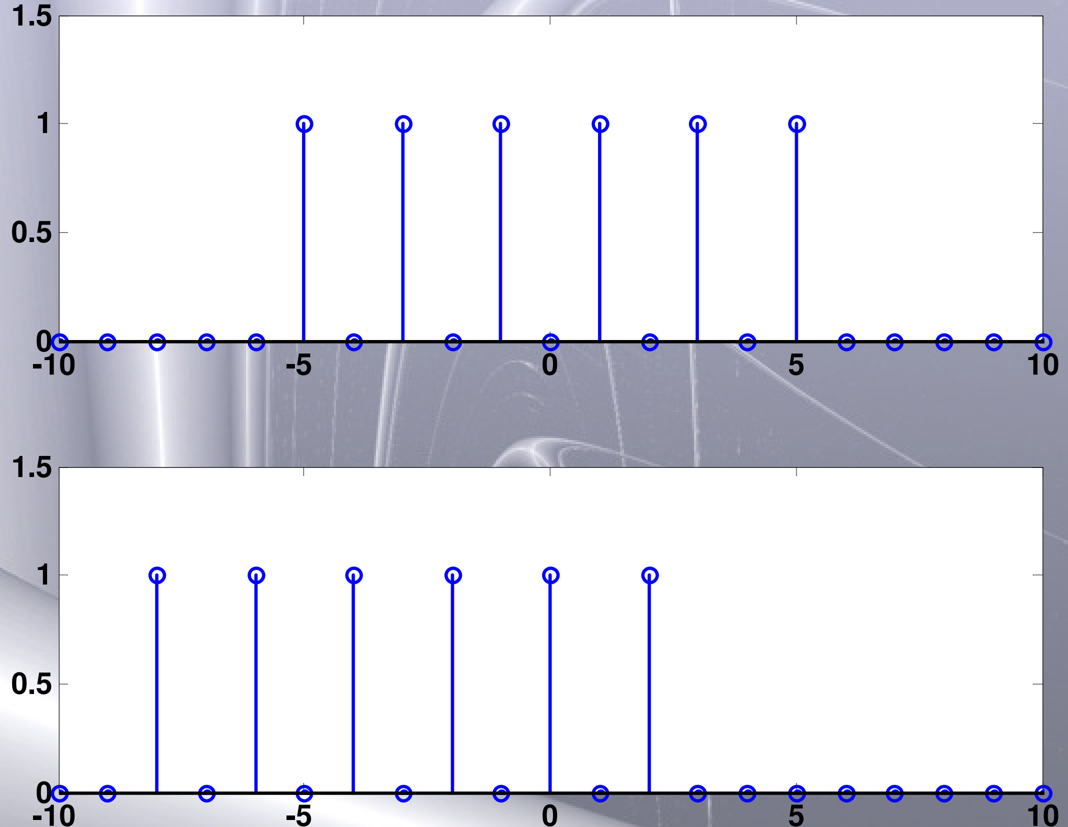
$x(2)x(4) = 0$

$x(3)x(5) = 1$

…

$\Sigma x(n)x(n+2) = 5$

Example 1: Energy-Defined Signal



…

x(-5)x(-2) = 0

x(-4)x(-1) = 0

x(-3)x(0) = 0

x(-2)x(1) = 0

x(-1)x(2) = 0

x(0)x(3) = 0

x(1)x(4) = 0

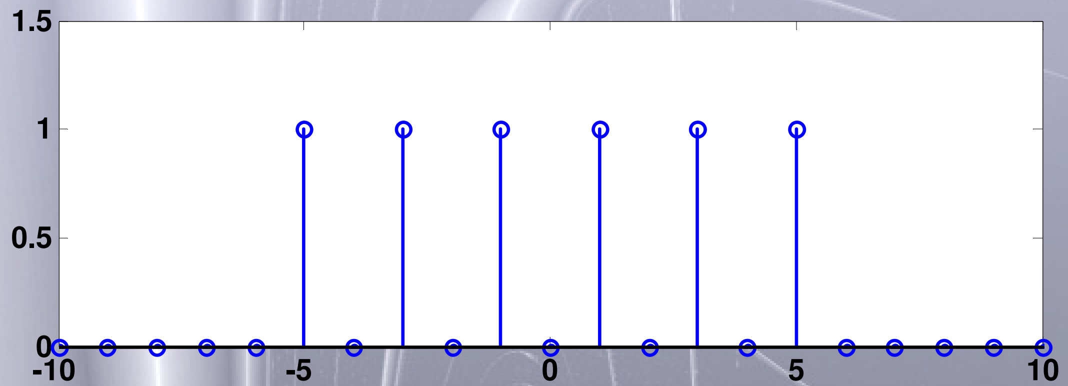
x(2)x(5) = 0

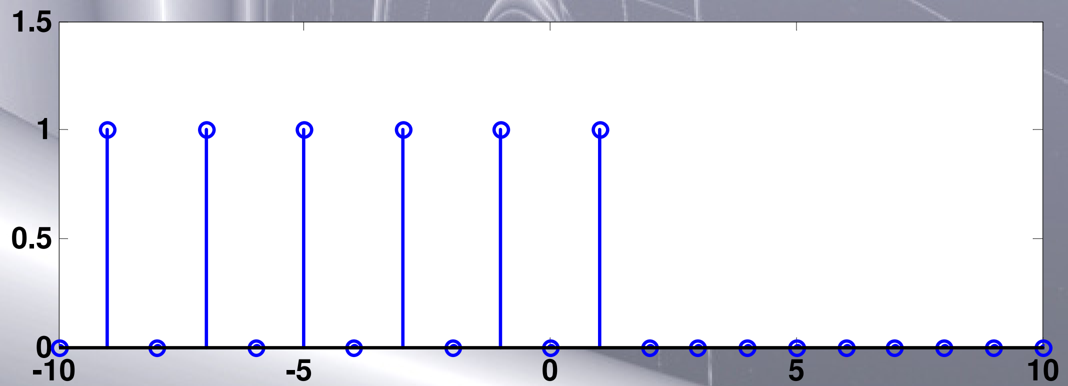
x(3)x(6) = 0

…

Σx(n)x(n+3) = 0

Example 1: Energy-Defined Signal





...

x(-5)x(-1) = 1

x(-4)x(0) = 0

x(-3)x(1) = 1

x(-2)x(2) = 0

x(-1)x(3) = 1

x(0)x(4) = 0

x(1)x(5) = 1

x(2)x(6) = 0

x(3)x(7) = 0

...

$\Sigma x(n)x(n+3) = 4$

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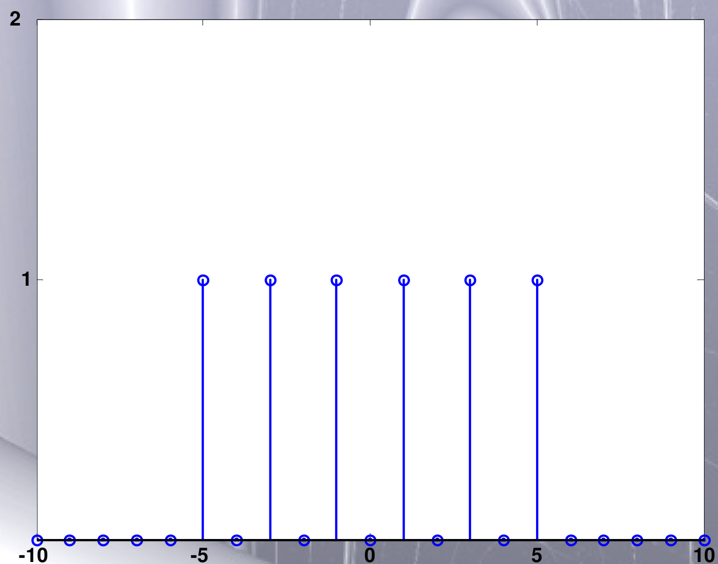
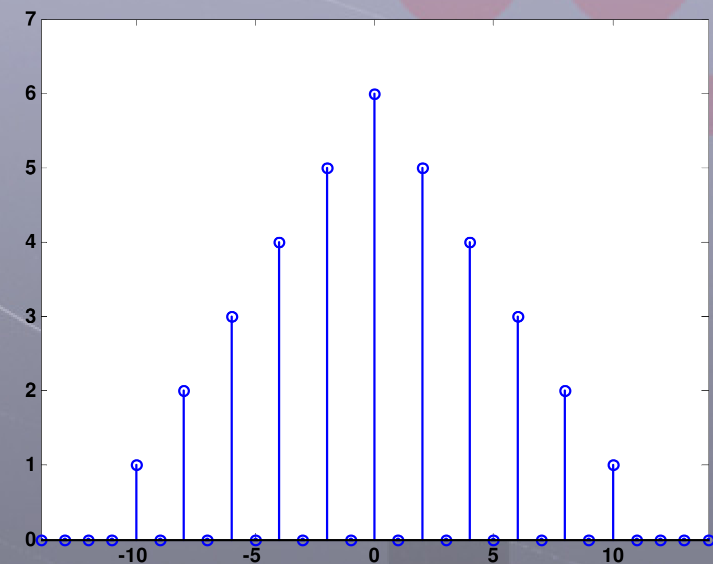
Example 1: Energy-Defined Signal



x(n)



$R_x(k)=\Sigma x(n)x(n+k)$

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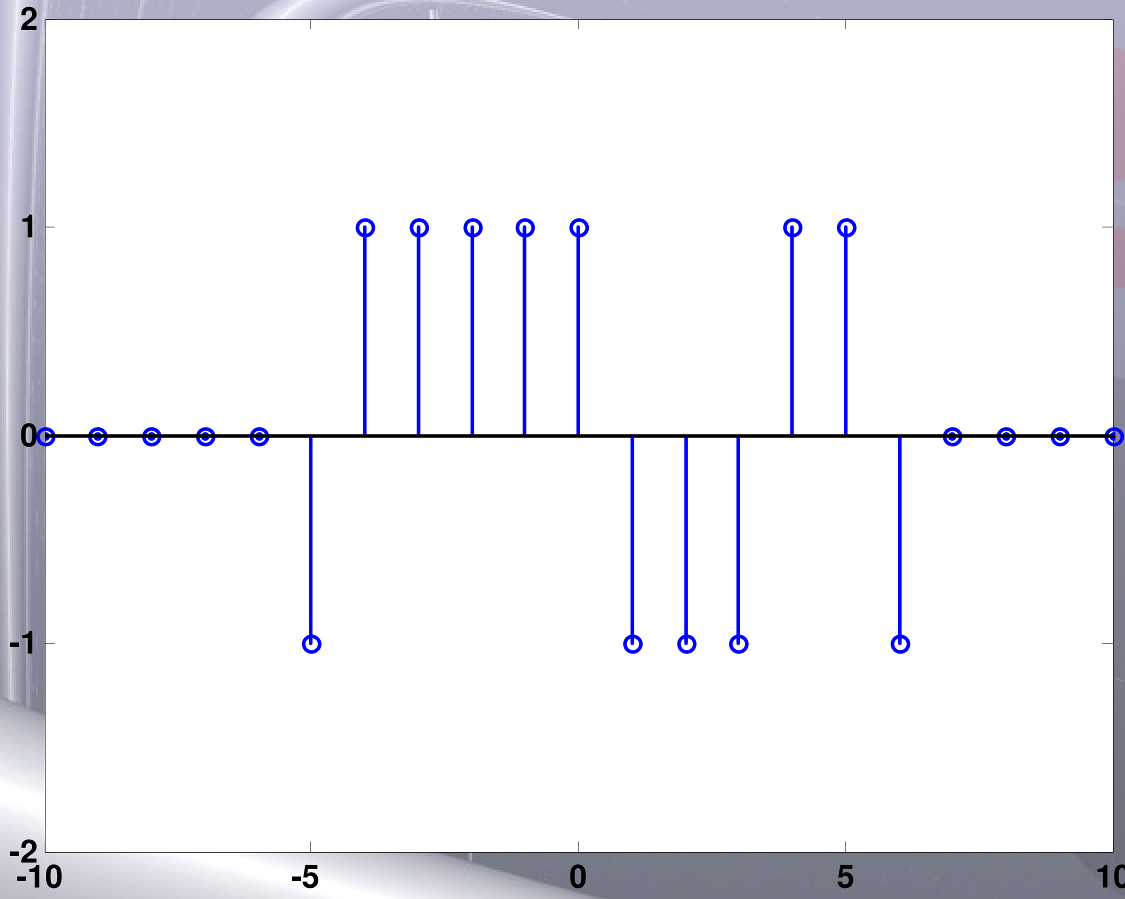
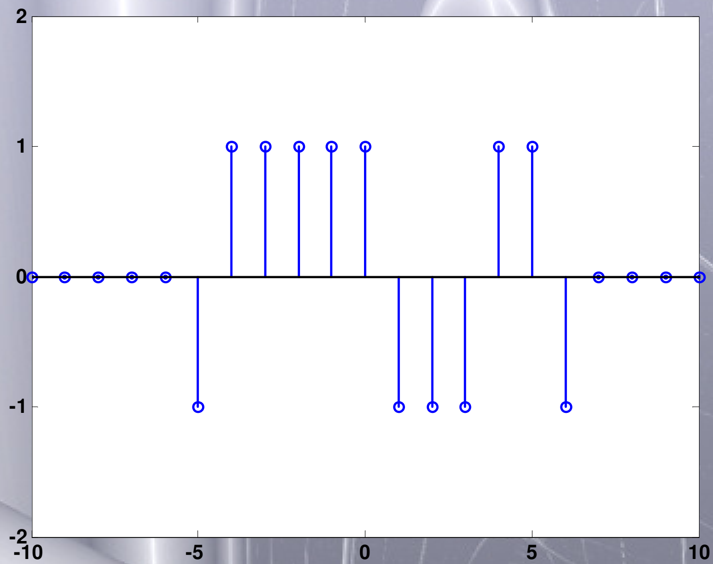
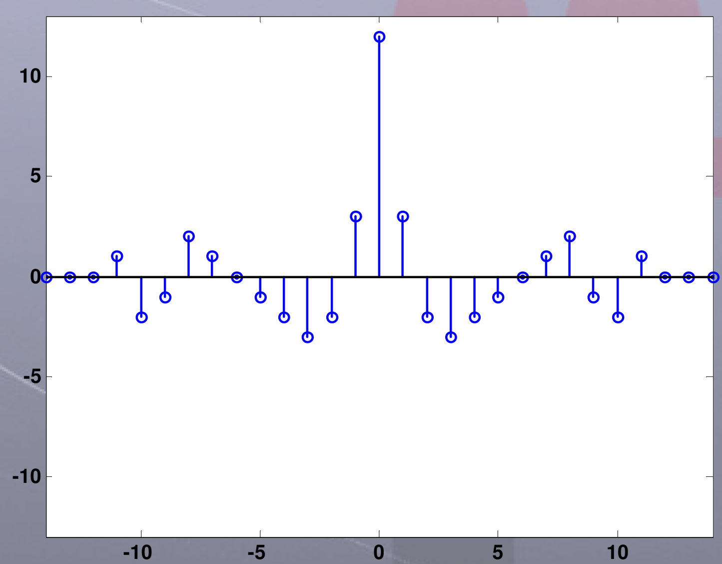
Example 2: Energy-Defined Signal



Example 2: Energy-Defined Signal



x(n)



$R_x(k)=\Sigma x(n)x(n+k)$

Autocorrelation for Energy-Defined Signals

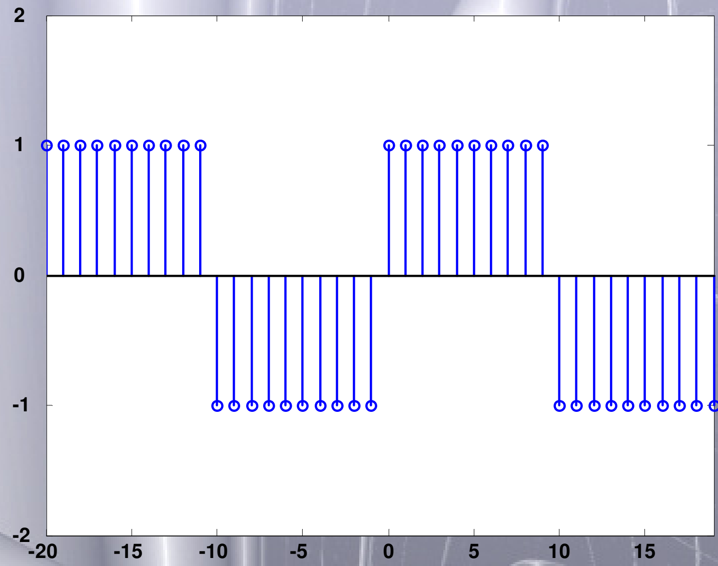
- If x[n] is a discrete signal energy-defined, its autocorrelation function $R_x[k]$ is defined as:

$$R_{\mathbf{x}}[k] = \sum_{n=-\infty}^{\infty} x[n]x[n-k]$$
$$= x[k] * x[-k]$$

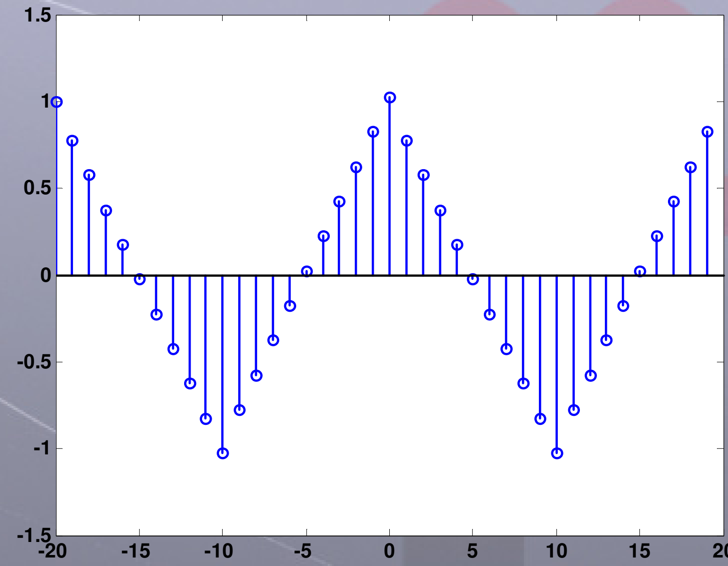
- If x(t) is a continuous-time signal energy-defined, its autocorrelation function $R_x(t)$ is defined as:

$$R_{\mathbf{x}}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$
$$= x(\tau) * x(-\tau)$$

Example 3: Power-Defined Signal



x(n)



$R_{\mathbf{x}}[k]$

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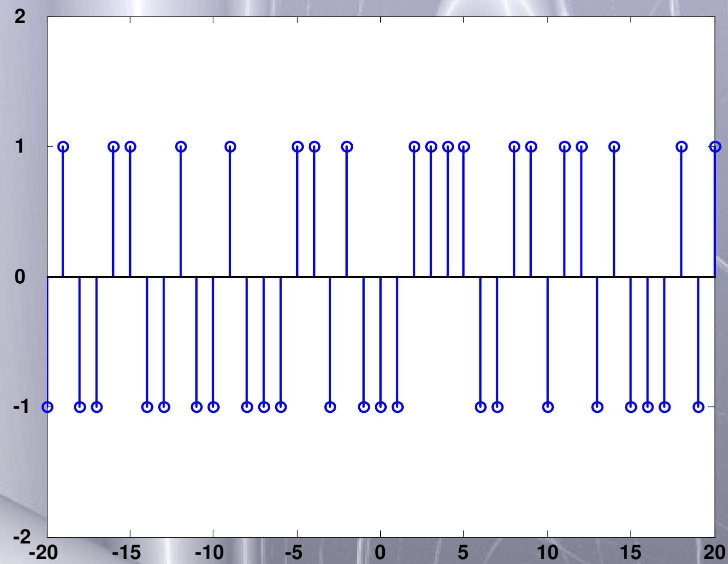
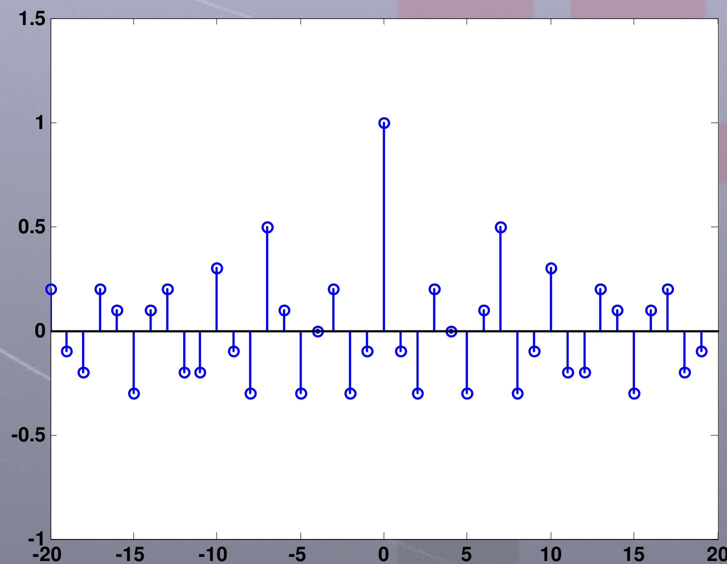
Example 4: Power-Defined Signal



x(n)



$R_x[k]$

Autocorrelation for Power-Defined Signals

- If x[n] is a discrete signal power-defined, its autocorrelation function $R_x[k]$ is defined as:

$$R_{\mathbf{x}}[k] = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x[n]x[n-k]$$

- If x(t) is a continuous-time signal power-defined, its autocorrelation function $R_x(t)$ is defined as:

$$R_{\mathbf{x}}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt$$

Homework

- Compute the autocorrelation of: $x(t) = A\cos(2\pi f t)$

Autocorrelation Summary

- The autocorrelation function is a measurement of the self-likeness of a signal – in other words, the autocorrelation gives information about the likeness of a signal with itself but delayed
- Therefore, the autocorrelation function summarizes the time behavior of a signal
- Mathematically, the autocorrelation function is a projection (inner product) of a signal against a delayed version of itself with all possible values of delay. Consequently, the maximum of the autocorrelation function is at delay equal to zero – the likeness of a signal with itself – its power or energy

Autocorrelation Properties

- Autocorrelation definition is different for energy-defined and power-define signals.

- In both cases, autocorrelation measures signal likeness to delayed version of itself, referring this likeness to its maximum value which happens at delay equal to zero.

- For energy-defined signals

$$R_{\mathbf{x}}(0) = E_x$$

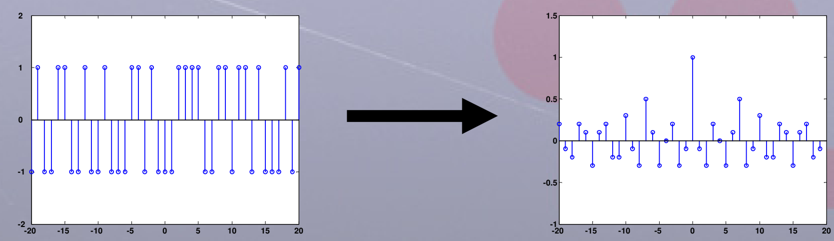
- And for power-defined signals

$$R_{\mathbf{x}}(0) = P_x$$

Autocorrelation Properties

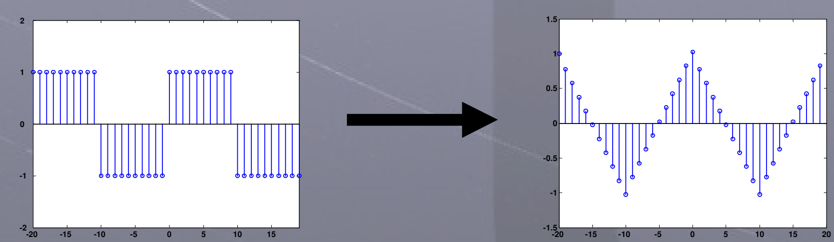
- Autocorrelation function satisfies:

$$\left|R_{\mathbf{x}}(\tau)\right| \leq R_{\mathbf{x}}(0)$$



- And for the particular case of periodic signals, it holds that:

$$R_{\mathbf{x}}(L \cdot T) = R_{\mathbf{x}}(0)$$



where L is any integer number and T is the signal period

Autocorrelation Properties

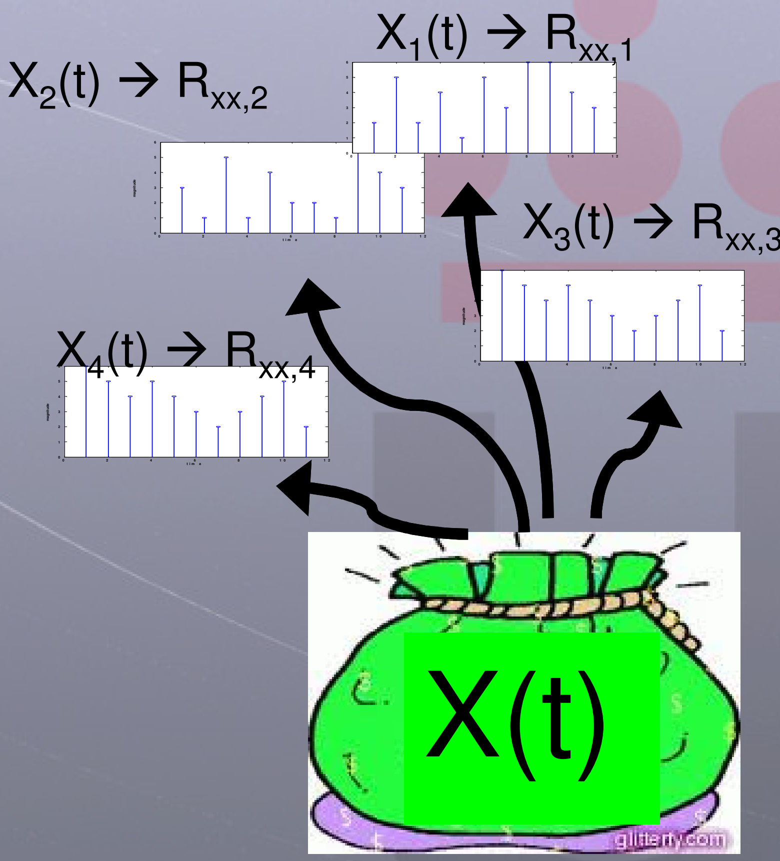
- Symmetry: $R_X(\tau)$ is and even signal: $R_X(\tau) = R_X(-\tau)$.
- Maximum: $R_X(\tau)$ maximum is for $\tau = 0$ (and coincides with energy/power of the signal), $|R_X(\tau)| \leq R_X(0)$.
- Periodicity: if for a give value of T, it holds that $R_X(T) = R_X(0)$, then it also holds that $R_X(kT+\tau) = R_X(0+\tau)$ for any integer value of k.
- Integrability: the autocorrelation function of any signal (except the periodic signals) can be integrated – i.e. it is a energy-defined signal itself.

Autocorrelation of Stochastic Processes

- Recall the two viewpoints for SP:
  - A set of signals with common properties, although signals itself are not identical point-by-point
  - A physical mechanism that generates sets of signals according to a stochastic pattern
- In any case, to describe an SP the set of signals has to be described, but a single signal can not be described.
- So, how can we define the autocorrelation of a SP?

Intuition

- Let's assume, as starting point, that we have a set of signals and we select one of them by a random mechanism.
- The autocorrelation of one of these realizations, $x_1(t)$, can be computed as its time average using next expression:



$$R_{\mathbf{xx},1}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x_1(t) x_1(t-\tau) dt$$

Intuition

- But if we compute autocorrelation in such a way, we are not computing the autocorrelation of the SP, but the one of a particular signal (realization)

- So, in order to compute the autocorrelation of the SP, we should average over all possible realizations:

$$
\begin{aligned}
R_{\mathbf{x}}(\tau) &= E\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)dt\right] \\
&= \lim_{T\to\infty} E\left[\frac{1}{2T}\int_{-T}^{T} x(t)x(t-\tau)dt\right] \\
&= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} E\left[x(t)x(t-\tau)\right]dt
\end{aligned}
$$

Intuition → Definition

- **If the expected value does not depend on the time**, then:

$$R_{\mathbf{x}}(\tau) = E\left[x(t)x(t-\tau)\right]\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}dt$$
$$= E\left[x(t)x(t-\tau)\right]$$

- The autocorrelation function for a SP is noted as $R_X(t_1,t_2)$, and its definition is

$R_X(t_1,t_2) = E\ [X(t_1)\ X(t_2)]$:

$$R_X(t_1,t_2) = E\left[X(t_1)X(t_2)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)\,dx_1 dx_2$$

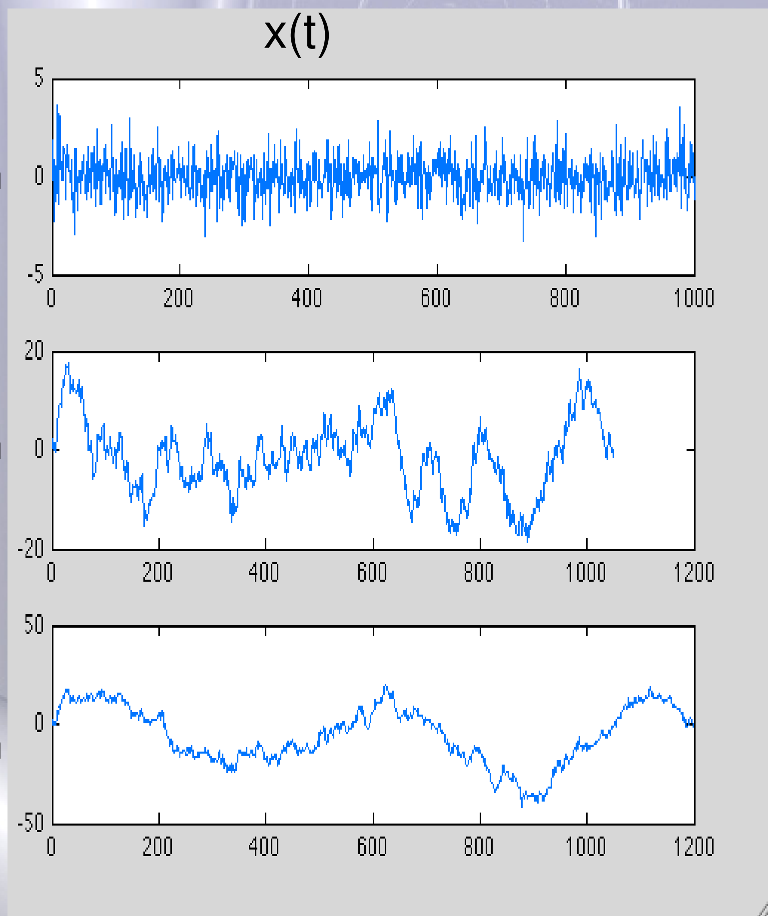
- Which is a measurement of the likeness between Random Variables obtained in two instants of the SP, $X(t_1)$ and $X(t_2)$. i.e. $R_X(t_1,t_2)$ is a measurement of the likeness (variation) of the signal in two different instants of time t_1 and t_2.

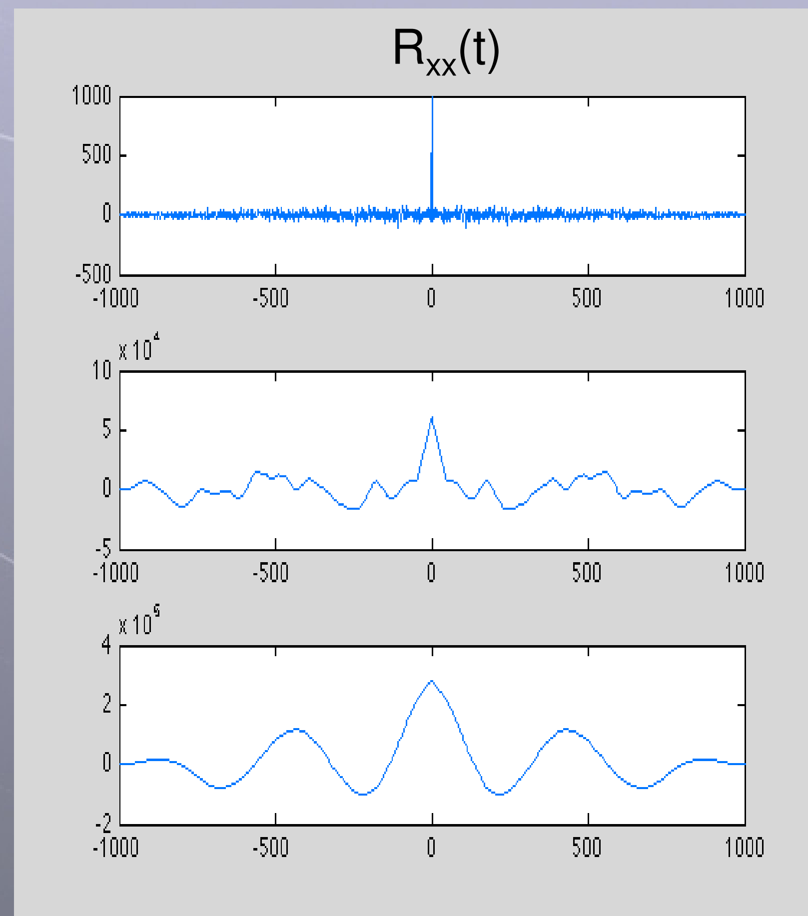
Example: Rxx based on one SP realization

One realization of SP 1

One realization of SP 2

One realization of SP 3



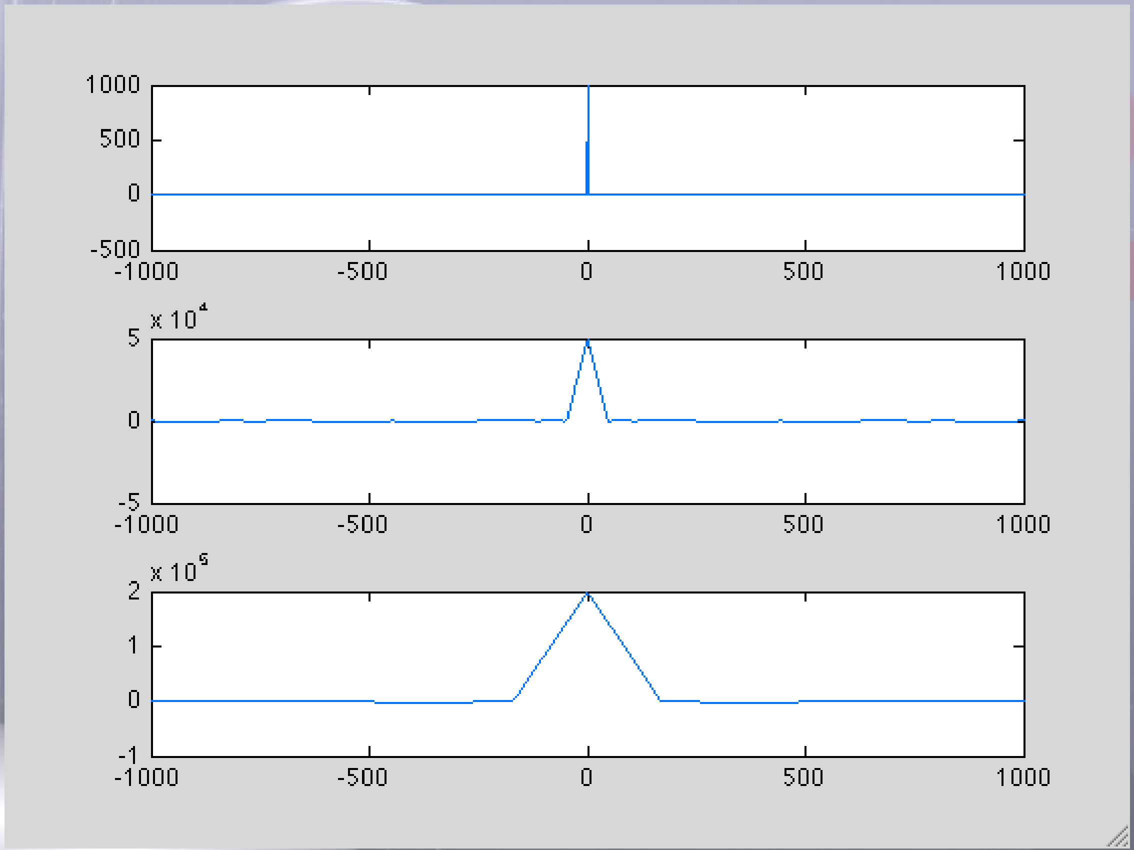


SP 3 more correlated than SP 2, which is more correlated than SP 1 (uncorrelated)

Autocorrelations obtained using only the single realization

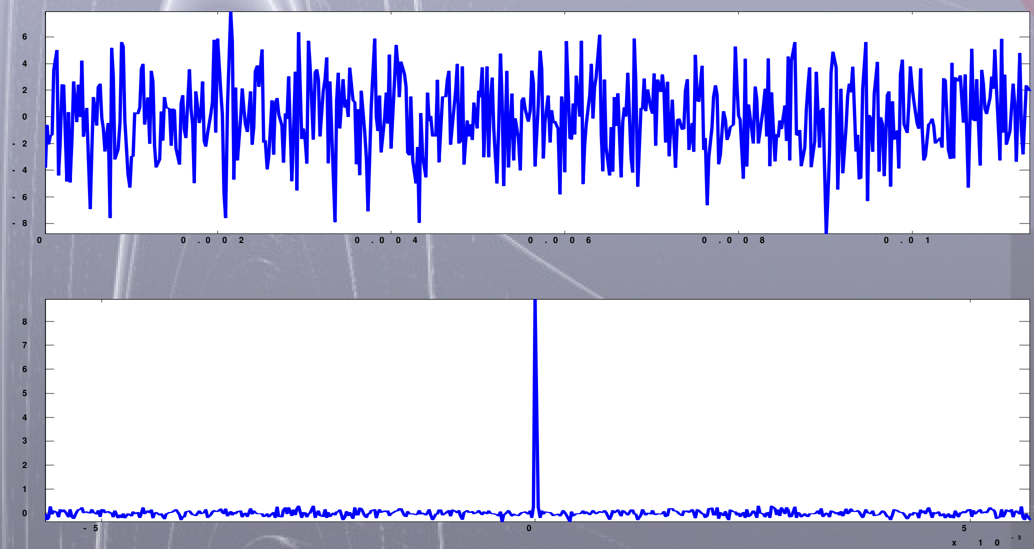
Example: Rxx averaging over “all” (many) SP realizations

SP 3 more correlated than SP 2, which is more correlated than SP 1 (uncorrelated)



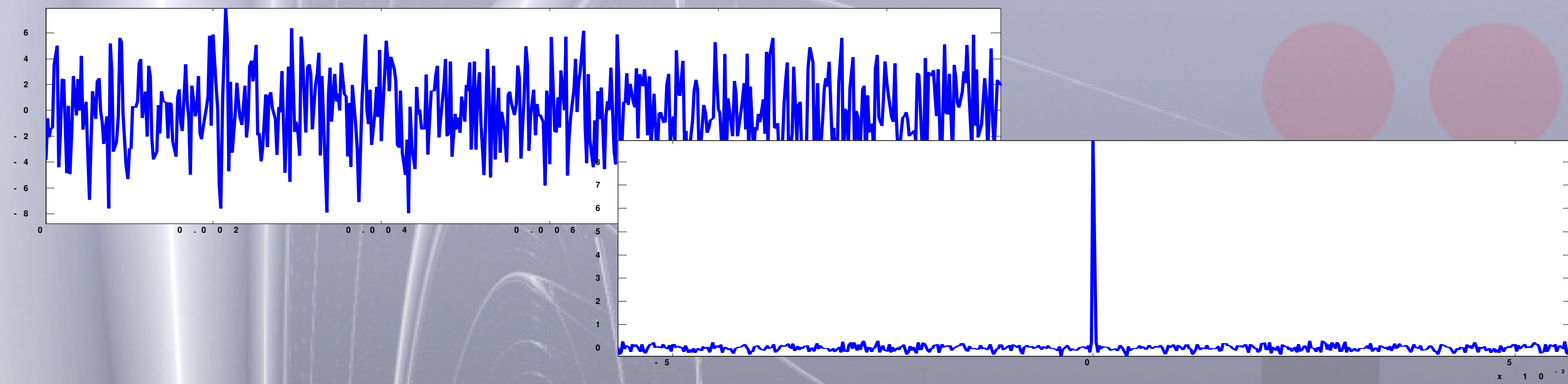
Uncorrelated Processes

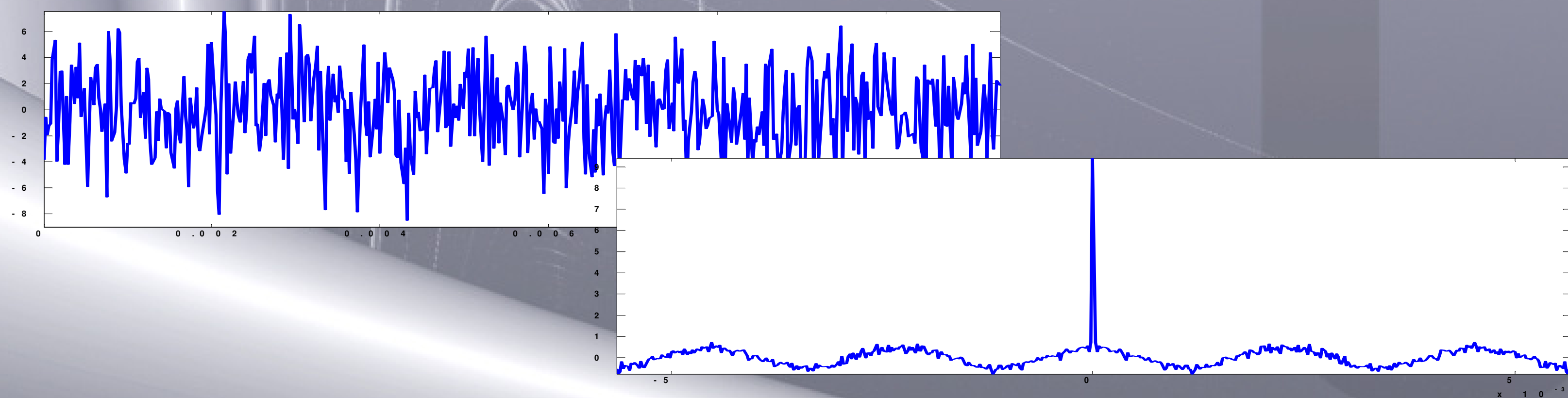
- A SP is uncorrelated if $R_X(t_1,t_2) = 0$, for any t_1 and t_2 such that $t_1 \neq t_2$.



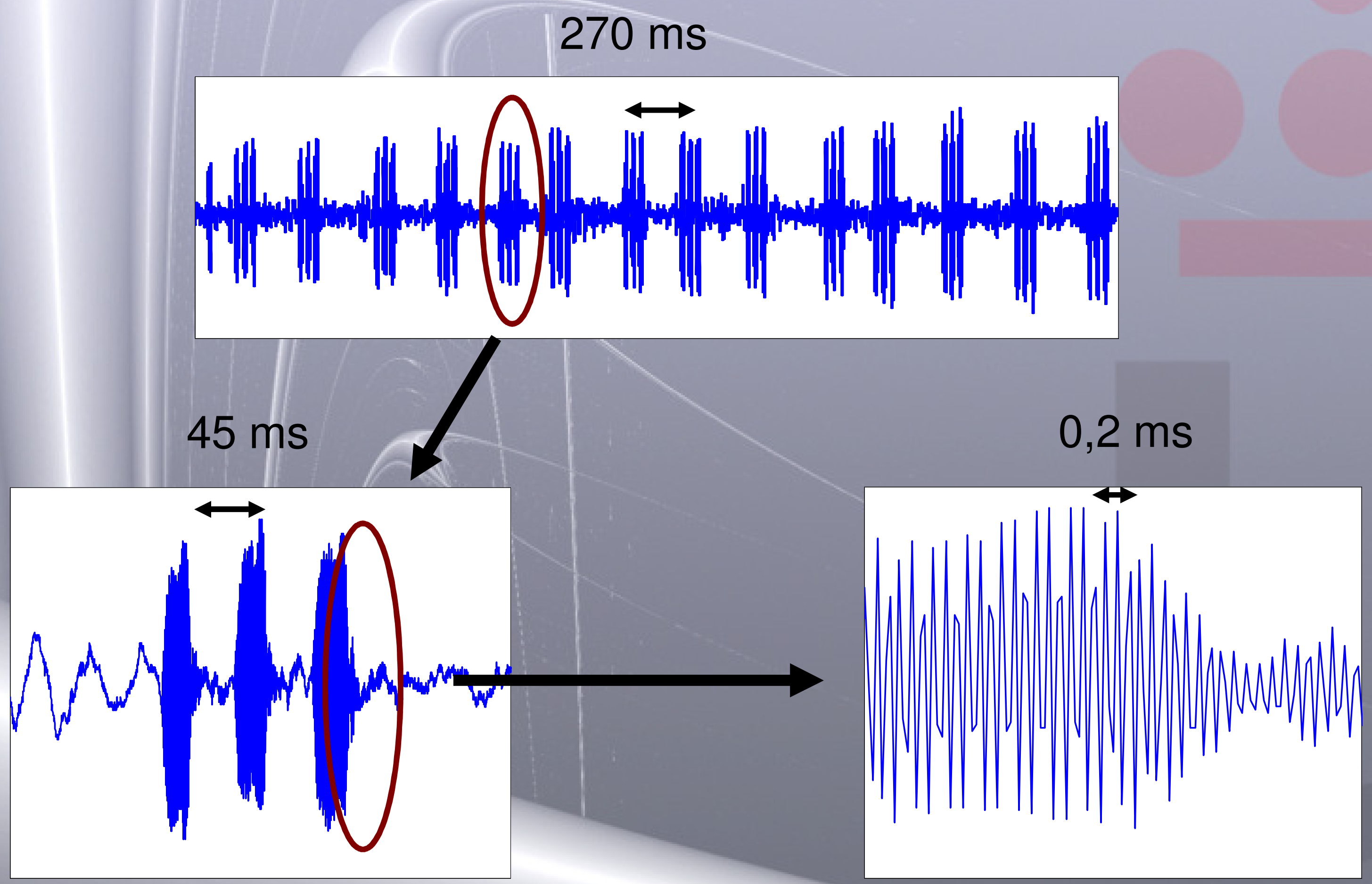
  – Intuitively, there is not “likeness” between samples at different times of the process

Noise in Telecommunications often is Uncorrelated

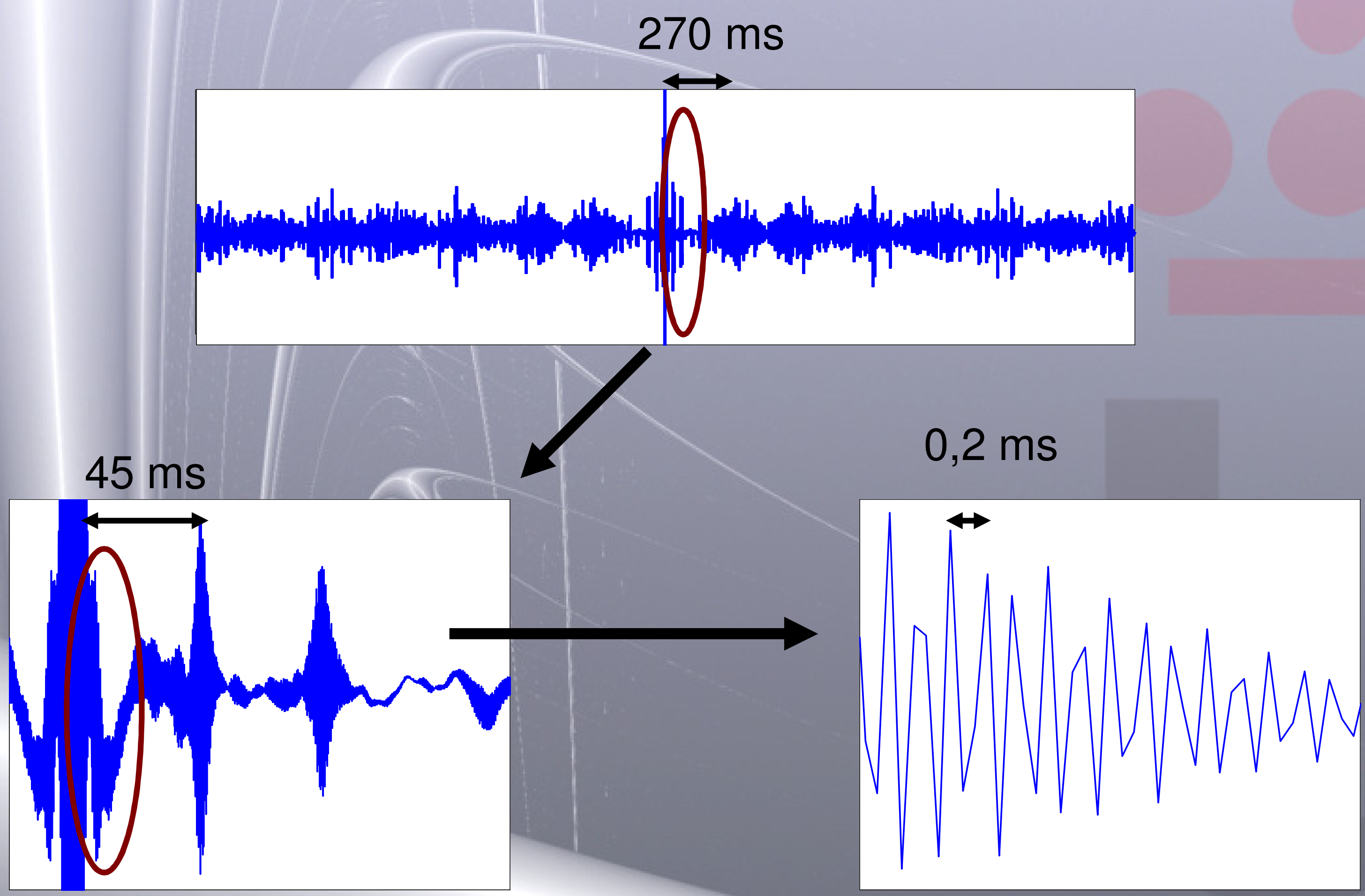




Sine on/off wave (cricket)



Sine on/off wave



Summarizing

- Autocorrelation is a measurement of the likeness between a signal and a delayed version of itself
- Each realization of a SP may be quite different and computing any statistic on it will mislead to totally incorrect information. Autocorrelation has to average all (many) realizations, i.e. compute the expected value
- Peak values in the autocorrelation function hints about repetition patterns

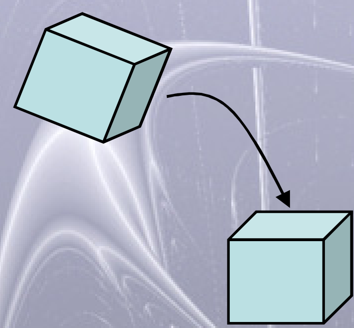
Statistical Independence

- Two scenarios:
  - Independence of the samples of a signal
  - Independence of two signals
- Intuitively, two samples are independent if the mechanisms that generates them are also independent
- Formally, two Random Variables, X_1 and X_2, are independent if, and only if,

$$f_{X_1X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2)$$

Intuition for Independent Samples

Rolling a dice

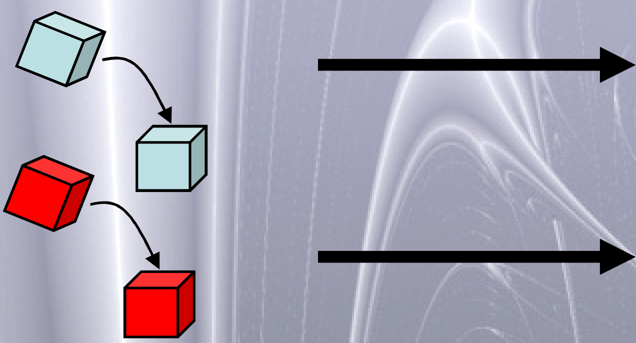


Independent: → 3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3, …

Dependent: → **4**, **6**, **6**, 5, 6, 4, 3, 7, 10, 7, …

Intuition for Independent Signals

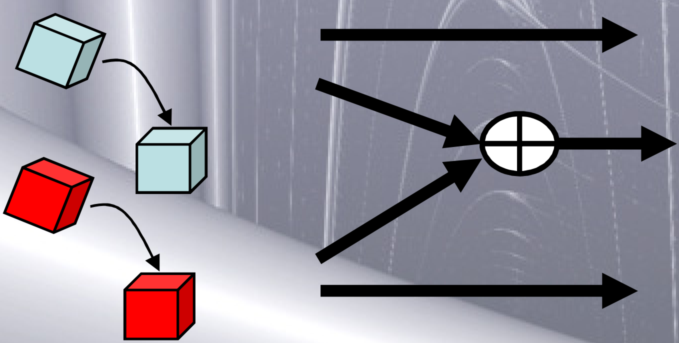


3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

1, 1, 6, 4, 3, 1, 2, 4, 5, 5, 2

Indepe
ndent?

Independent:



3, 1, 5, 1, 4, 2, 2, 1, 6, 4, 3

4, 2, 11, 5, 7, 3, 4, 5, 11, 9, 5

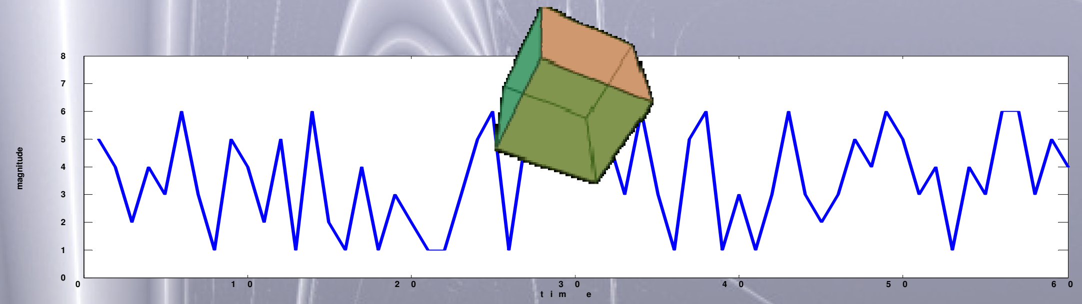
1, 1, 6, 4, 3, 1, 2, 4, 5, 5, 2

Indepe
ndent?

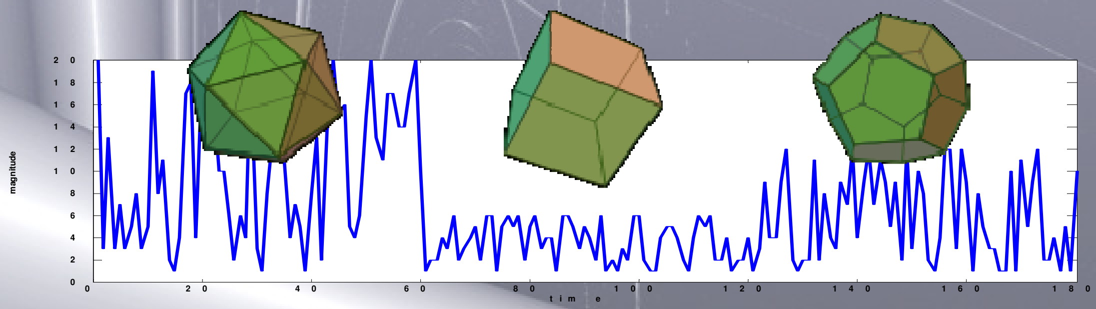
Dependent:

SP and Stationarity

- An SP is stationary if its statistics do not depend on time



Stationary



Non-Stationary

Stationary Stochastic Processes

- When arriving to practical SP, two types of stationarity can be defined:
  - Strict (Sense) Stationary Processes (SSP). The pdf of the process does not change with time delay Δ

$$f_X(t_1, t_2, ..., t_n) = f_X(t_1 + \Delta, t_2 + \Delta, ..., t_n + \Delta)$$

  - Wide Sense Stationary Processes (WSSP), those that satisfy the two following restrictions:
    - The mean value of the process, E{X(t)}, does not varies with time

$$R_X(t_1, t_2) = E\left[X(t_1)X(t_2)\right]$$

    - The autocorrelation $R_X(t_1,t_2)$ depends only on the time difference $\tau = t_1 - t_2$. Thus, for WSSP we compute the autocorrelation as $R_X(\tau)$.

$$R_{\mathbf{x}}(\tau) = E\left[x(t)x(t-\tau)\right]$$

- An SSP is also WSSP, but the opposite is not true

SP and Ergodicity

- A Stochastic Process is Ergodic if its statistics can be computed as time average of one of its realizations

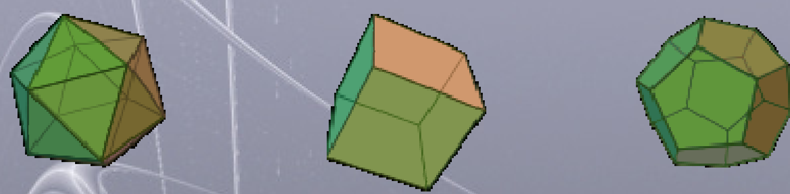
$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g(x(t))dt = E(g(X(t)))$$

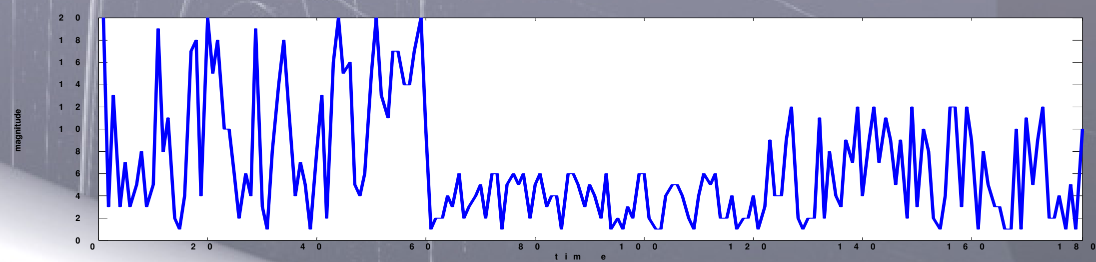
- Ergodicity implies stationarity. Any ergodic process is stationary
- However, stationarity does not imply ergodicity

- Example: roll a dice infinite times

Example 7

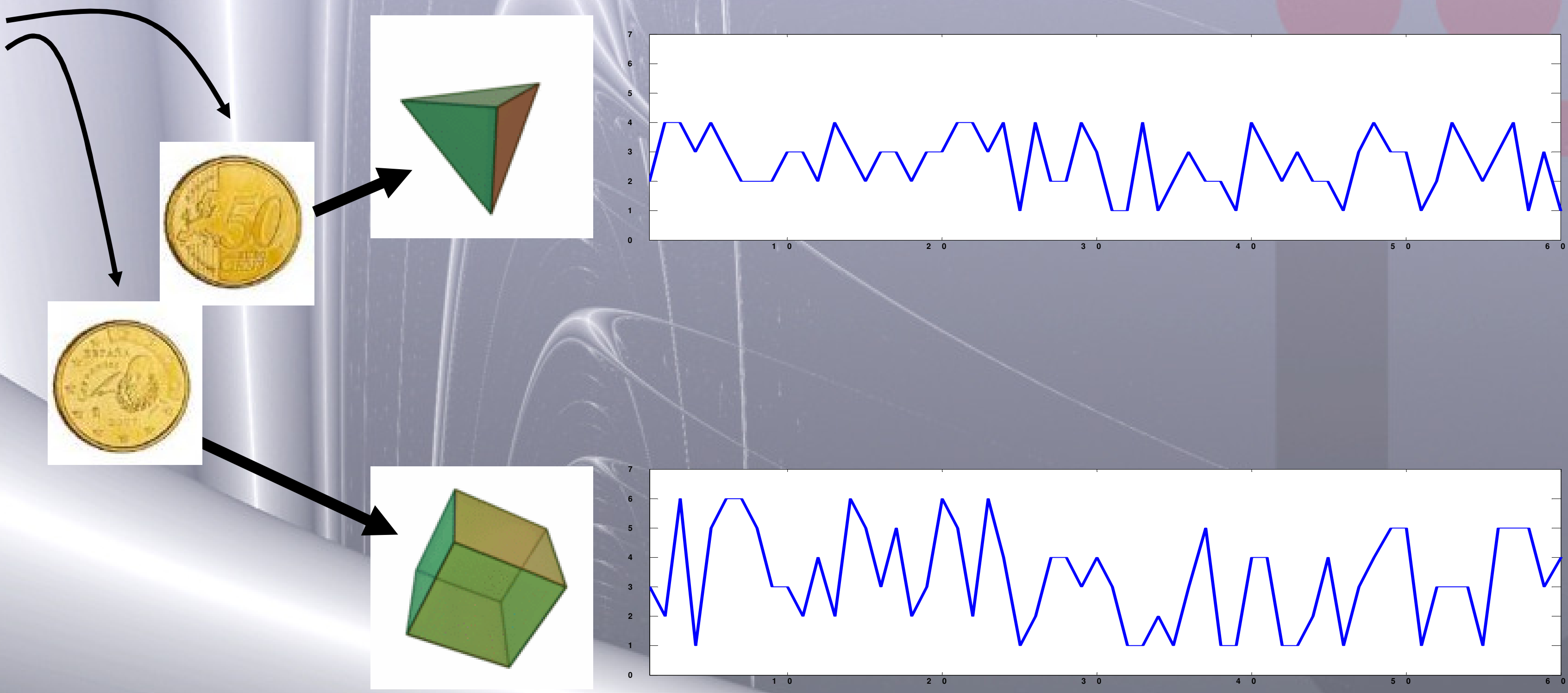
- The following SP is not stationary, and therefore it is not ergodic either.





Example 8

- The following Strict Sense Stationary Process is not Ergodic. Why?



Time Averaging and Statistical Expected Value

- Time average and expected value returns the same value only for ergodic SP

- As example, if we assume that human voice signal corresponds to a ergodic SP, then we can estimate statistics from time averaging only one recorded piece of voice, and assume that those values coincide with the statistics of any voice signal

Time Averaging and Statistical Expected Value

$$\langle x(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,dt$$

$$\Updownarrow$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$$

$$\langle x^2(t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x^2(t)\,dt$$

$$\Updownarrow$$

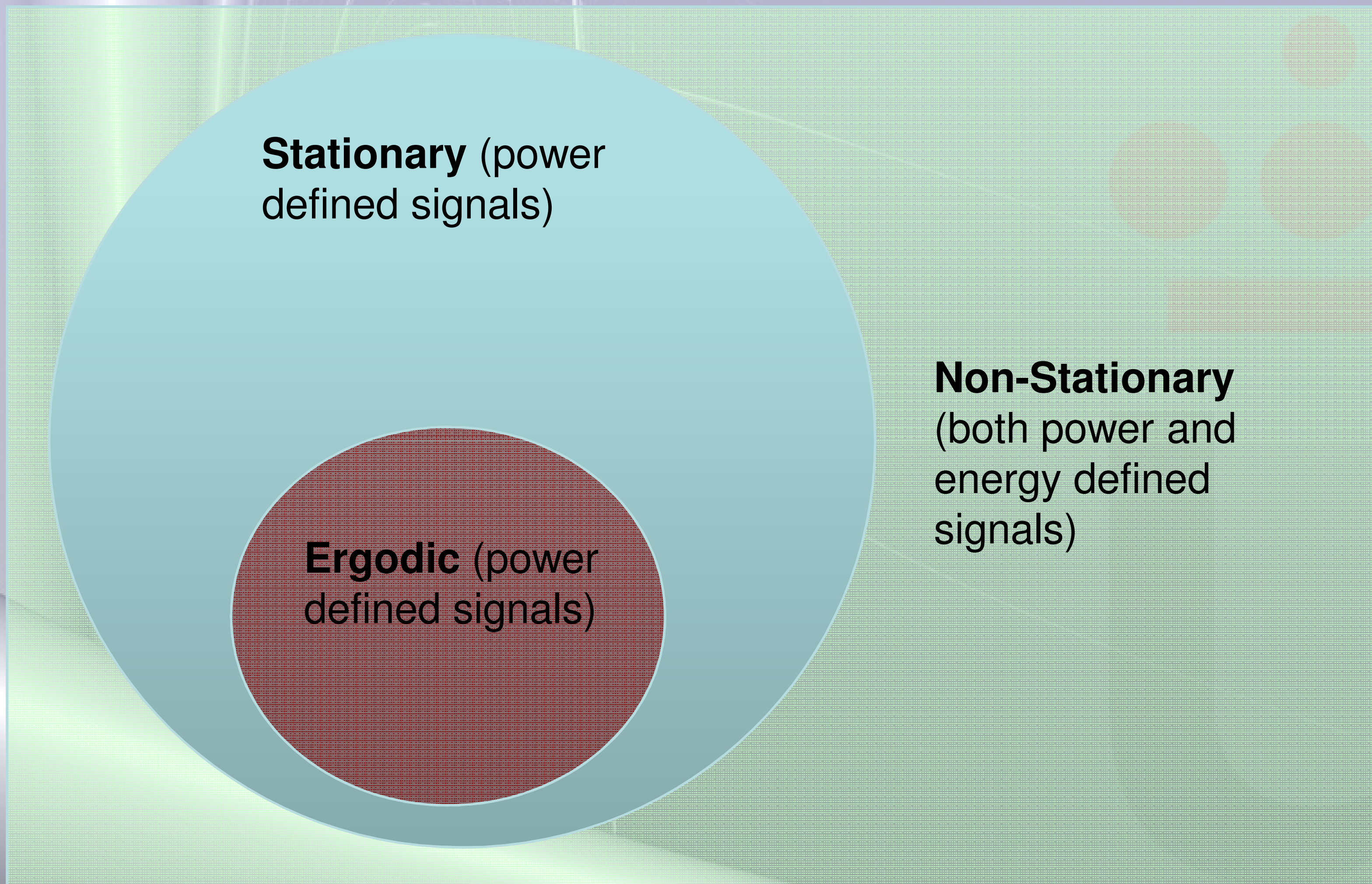
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$R_{\mathbf{x}}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)\,dt$$

$$\Updownarrow$$

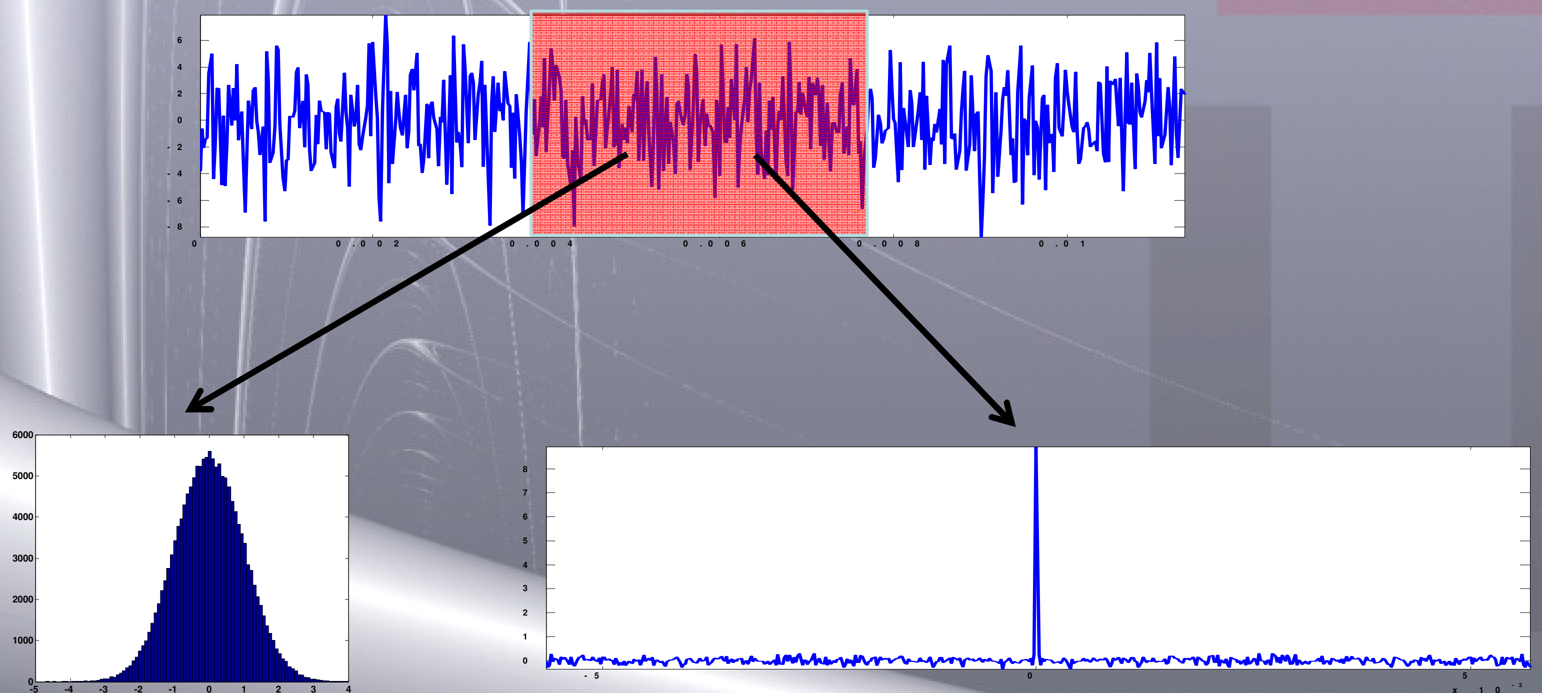
$$R_X(\tau) = E(X(t)X(t-\tau))$$

SP classification



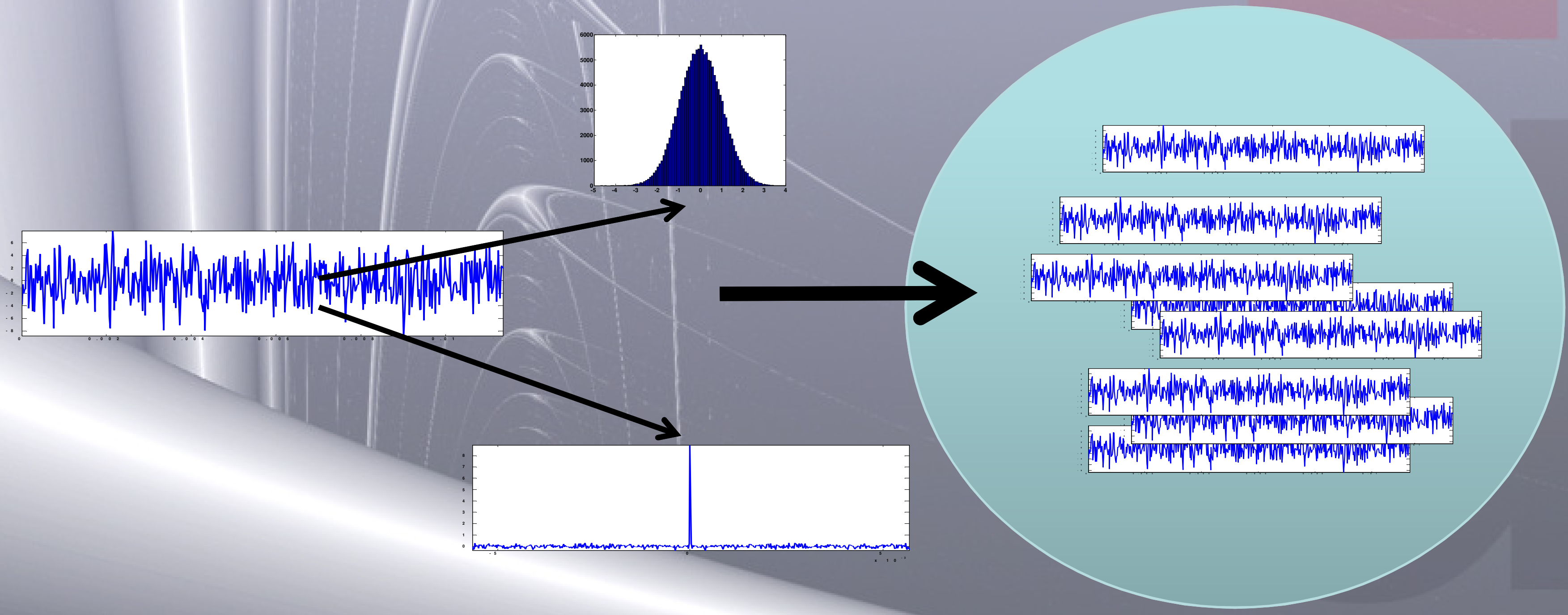
What is the practical meaning of Stationarity?

- We can characterize one realization from a segment of it



What is the practical meaning of Ergodicity?

- We can characterize the SP from a realization of it



Cross-Correlation

- If autocorrelation is a measurement of how a signal looks like itself delayed; the cross-correlation function provides information about likeness of two signals (delaying one respect the other)
- Depending of the type of signals, cross-correlation can be defined for:
  - Cross-Correlation of Energy Defined signals
  - Cross-Correlation of Power Defined signals
  - Cross-Correlation of one Energy Defined signal and one Power Defined signal

Cross-Correlation of two Energy Defined Signals

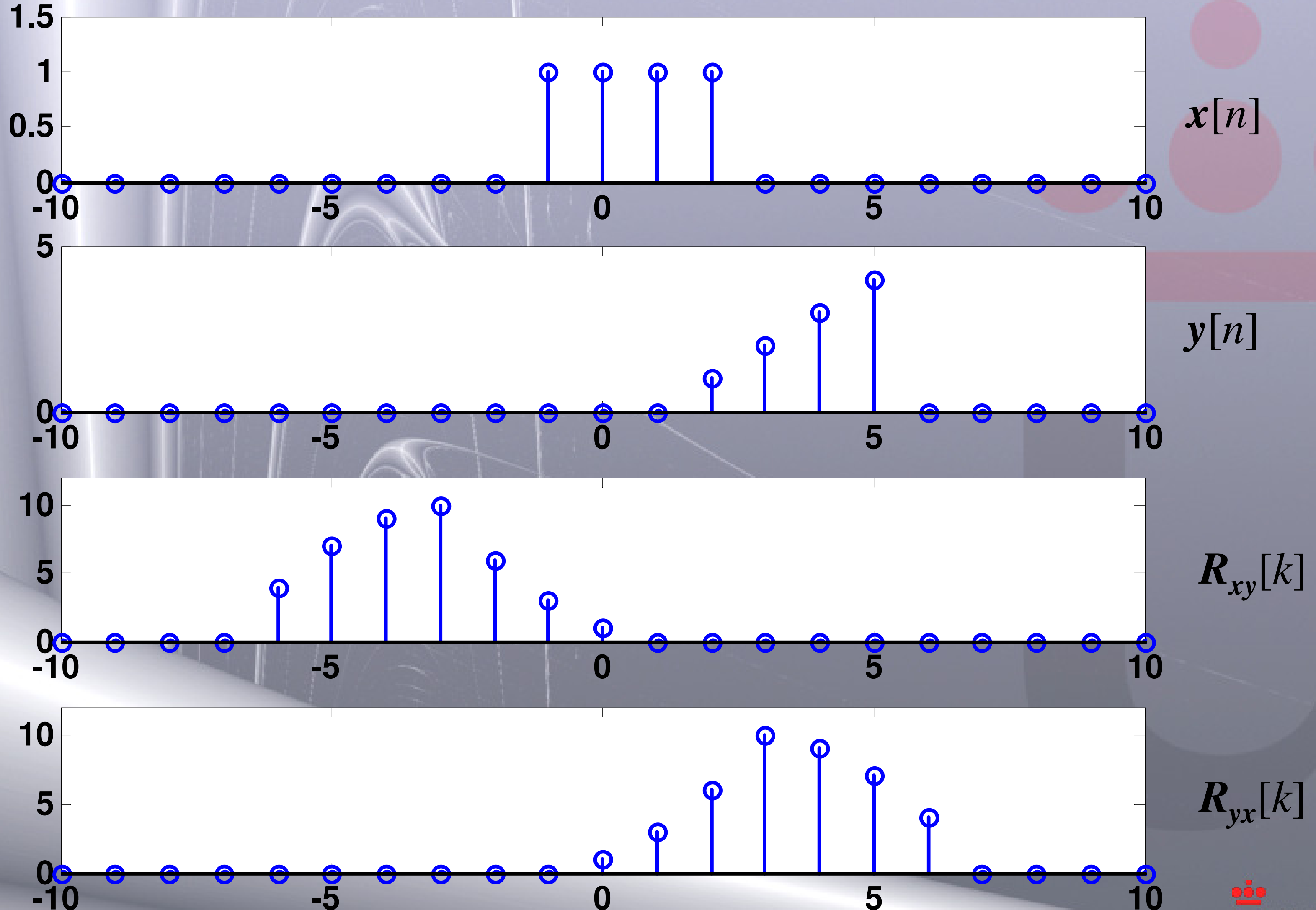
- Let x[n] and y[n] be two energy defined discrete signals (**or one Energy defined and the other Power defined).** Their cross-correlation function is defined as:

$$R_{\mathbf{xy}}[k] = \sum_{n=-\infty}^{\infty} x[n]y[n-k]$$
$$= x[k] * y[-k]$$

- If x(t) and y(t) are two energy defined continuous signals (**or one Energy defined and the other Power defined).** Their cross-correlation function is defined as:

$$R_{\mathbf{xy}}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt = x(\tau) * y(-\tau)$$

Example 1



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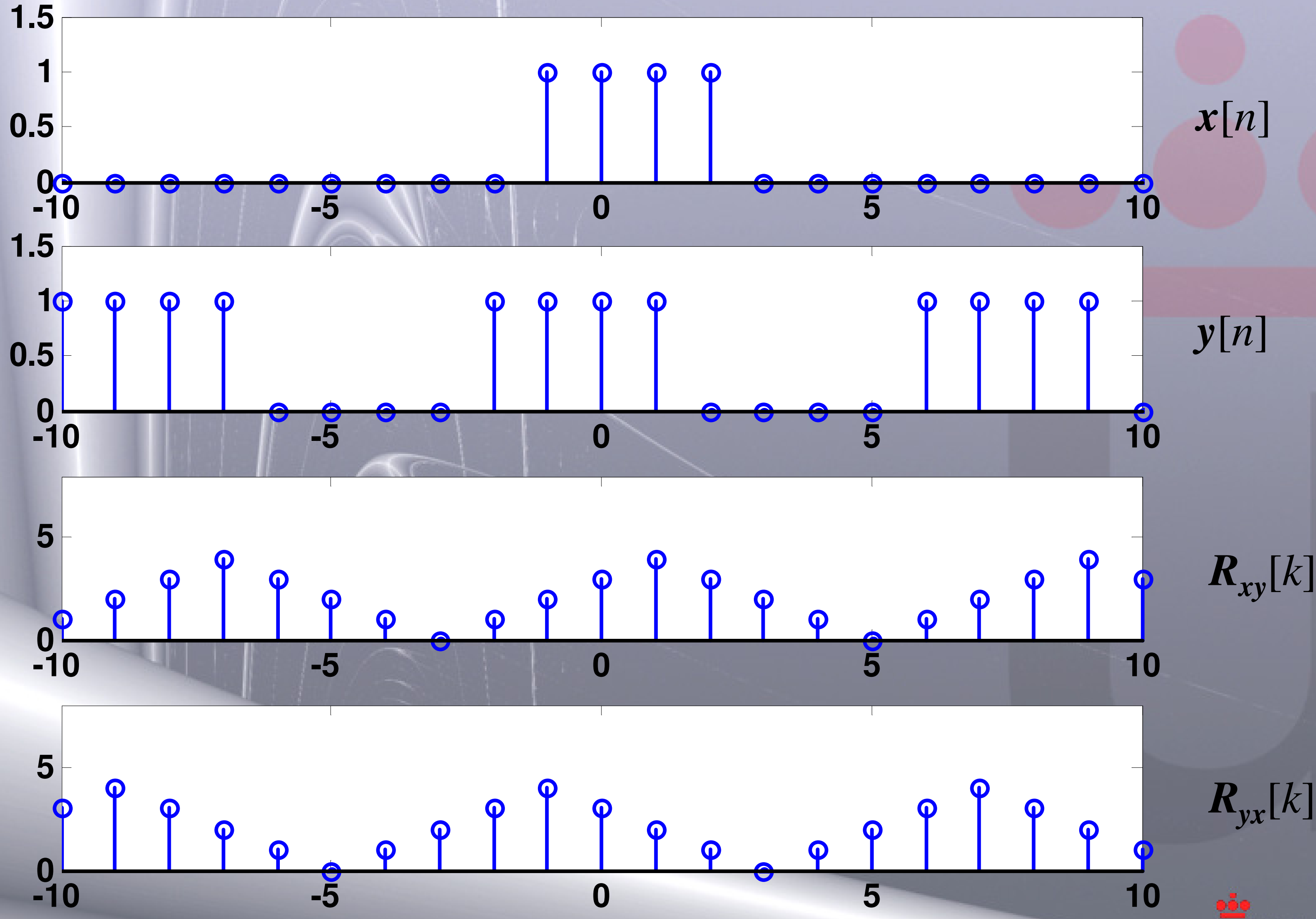
Example 2



Cross-Correlation of two Power Defined Signals

- Let x[n] and y[n] be two power defined discrete signals. Their cross-correlation function, $R_{xy}[k]$, is defined as:

$$R_{xy}[k] = \lim_{n \to \infty} \frac{1}{2n+1} \sum_{n=-\infty}^{\infty} x[n] y[n-k]$$

- Let $x(t)$ and $y(t)$ be two power defined continuous signals. Their cross-correlation function, $R_{xy}(\tau)$, is defined as:

$$R_{xy}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) y(t-\tau) dt$$

Cross-Correlation from a Statistical Viewpoint

- Cross-Correlation of two stochastic processes can be defined also using their joint-pdf:

$$R_{XY}(t_1,t_2) = E\left[X(t_1)Y(t_2)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),Y(t_2)}(x,y)\,dxdy$$

- When both processes are stationary, then it holds that:

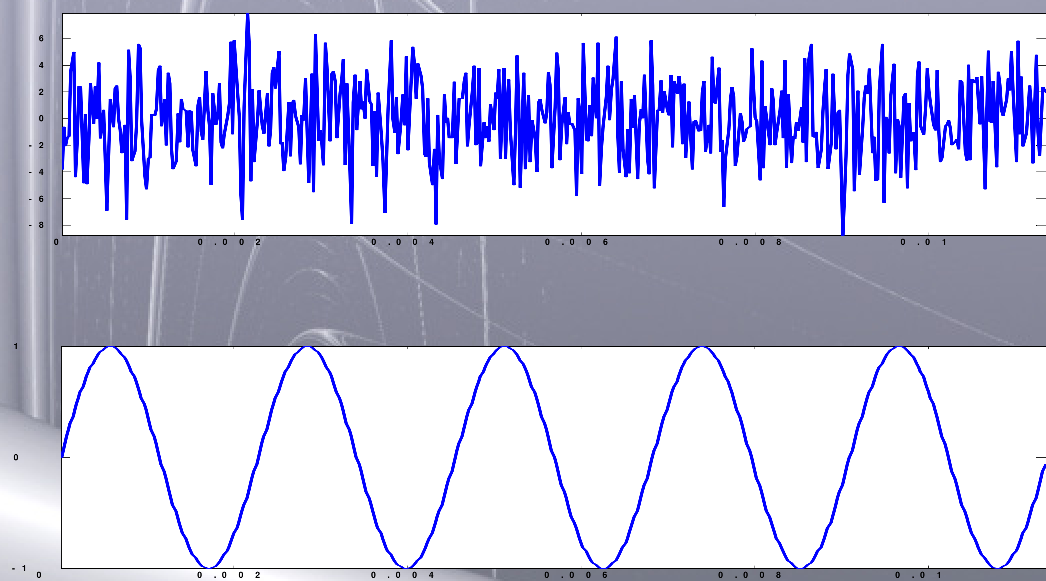
$$R_{XY}(\tau) = E\left[X(t)Y(t-\tau)\right]$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t),Y(t-\tau)}(x,y)\,dxdy$$

Cross-Correlation Properties

- $R_{xy}(\tau) = R_{yx}(-\tau)$.
- $P_{xy} = R_{xy}(0) = R_{yx}(0)$ can be understood as the cross-power between x(t) and y(t).
- The maximum of $R_{xy}(\tau)$ points to the time delay at which both signals exhibit their maximum likeness

Uncorrelated Processes

- Two SP are uncorrelated if $R_{xy}(\tau) = 0$ for every value of τ
- It can be proven that if two SP are independent and at least one of then has zero mean, then they are uncorrelated



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Sum of Signals

- Autocorrelation of the sum of two signal can be expressed as:

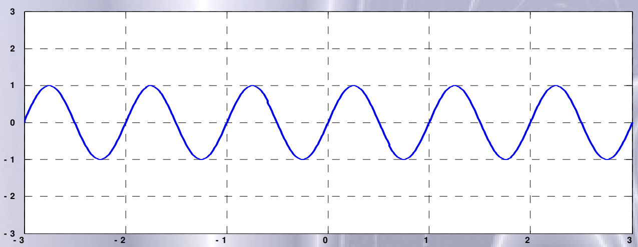
$$R_{\mathbf{x}+\mathrm{y}}(\tau) = R_{\mathbf{x}}(\tau) + R_{\mathrm{y}}(\tau) + R_{\mathbf{x}\mathrm{y}}(\tau) + R_{\mathrm{yx}}(\tau)$$

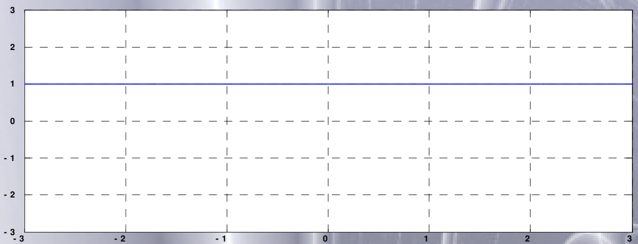
And particularizing for τ=0, the power of the sum is:

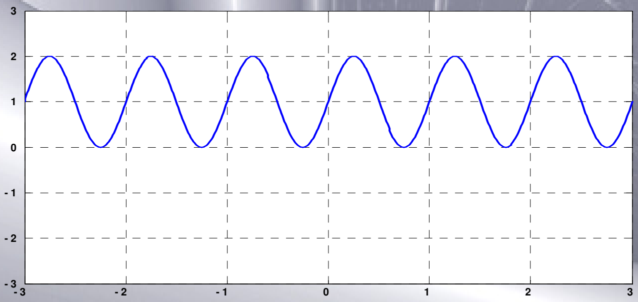
$$\begin{aligned} P_{\mathbf{x}+y} &= R_{\mathbf{x}+\mathrm{y}}(0) \\ &= R_{\mathbf{x}}(0) + R_{\mathrm{y}}(0) + R_{\mathbf{x}\mathrm{y}}(0) + R_{\mathrm{yx}}(0) \\ &= P_{\mathbf{x}} + P_{\mathrm{y}} + P_{\mathbf{x}\mathrm{y}} + P_{\mathrm{yx}} \end{aligned}$$

Important to note that only when the two processes are uncorrelated, the power of the sum is the sum of the powers

Sine Signal plus DC







$$P_{\sin} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2 \sin^2(ft)\,dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[-\frac{1}{2}A^2\cos(2ft) - \frac{1}{2}A^2\cos(ft - ft)\right]dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}2T\frac{1}{2}A^2 = \frac{A^2}{2}$$

$$P_{con} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} B^2 dt = B^2$$

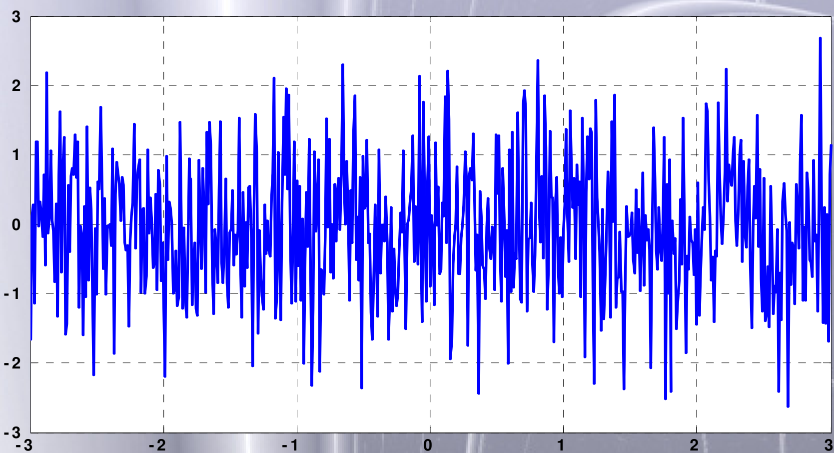
$$P_{con+\sin} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(B + A\sin(ft))^2\,dt$$

$$= \frac{A^2}{2} + B^2$$

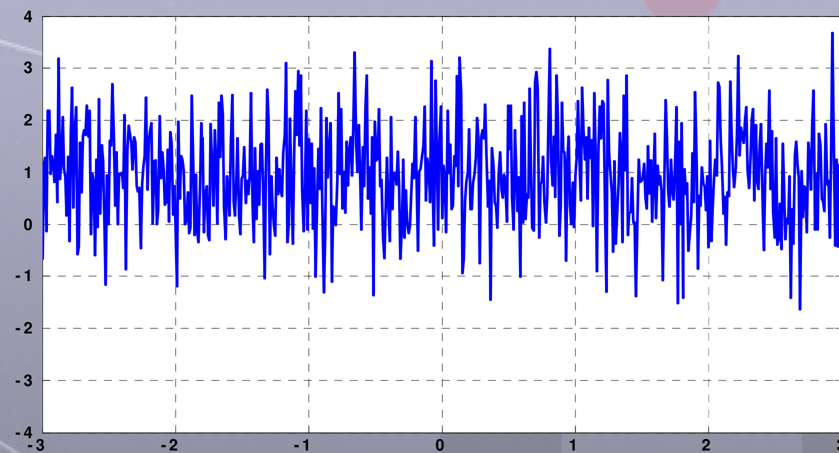
$$= P_{con} + P_{\sin}$$

Gaussian Noise

Zero mean



B mena



$$P_{noise} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \sigma^2$$

$$P_{noise+B} = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \sigma^2 + B^2 = P_{noise} + P_{con}$$

Why do we want to study the Spectrum?

- Cyclic processes are quite common in nature
  - Astronomy: lunar phases, planets orbits, solar storms, …
  - Biology: heart beat,…
  - Physics: acoustic vibrations, electromagnetic waves, …
- In communications, spectrum of the signal is of capital importance when designing and analyzing systems
- If a signal is deterministic, the Fourier Transform computes the spectrum, but what happen for Stochastic Processes?

Energy Spectral Density

- The Energy Spectral Density (ESD) of a energy-defined signal is calculated as the Fourier Transform of its auto-correlation function:

$$
\begin{aligned}
G_X(f) &= F\{R_X(\tau)\} \\
&= F\{x(\tau) * x(-\tau)\} \\
&= |X(f)|^2
\end{aligned}
$$

  - For Discrete time signals ESD is defined in similar way

Energy Spectral Density

- The Energy Spectral Density describes how the energy is distributed along different frequencies
- So, the total signal energy can be calculated by integrating the ESD for all frequencies

$$E_X = \int_{-\infty}^{\infty} |x(t)|^2 dt$$
$$= \int_{-\infty}^{\infty} |X(f)|^2 df$$
$$= \int_{-\infty}^{\infty} G_X(f) df$$

Power Spectral Density

- The Power Spectral Density (PSD) of a power-defined signal is calculated as the Fourier Transform of its auto-correlation function:

$$S_X(f) = F\{R_X(\tau)\}$$

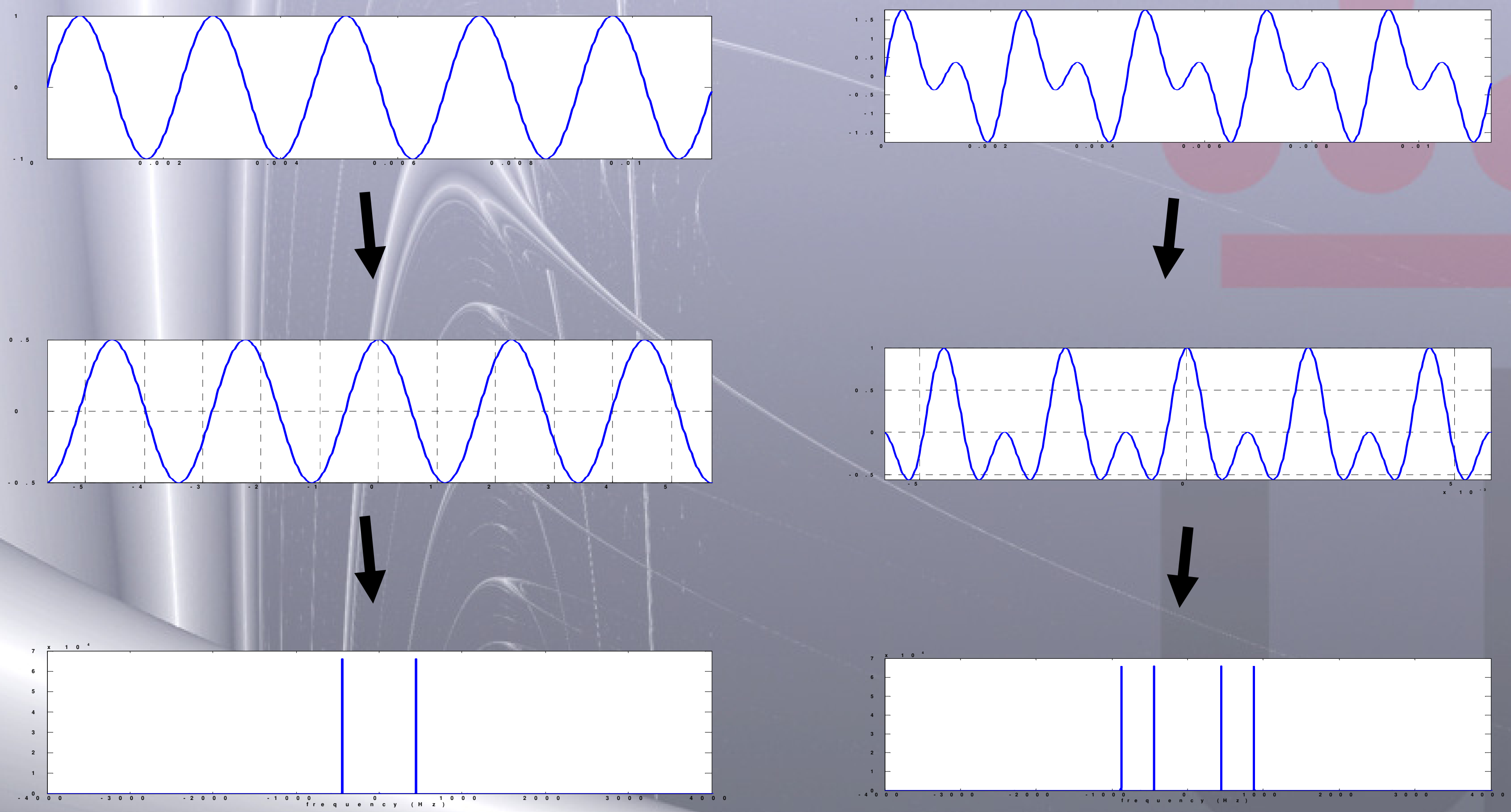
  - For Discrete time signals PSD is defined in similar way

Power Spectral Density

- The Power Spectral Density describes how the power is distributed along different frequencies
- So, the total signal power can be calculated by integrating the PSD for all frequencies

$$P_X = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$$
$$= \int_{-\infty}^{\infty} S_X(f) \, df$$

Sine Wave and the Sum of two Sine Waves



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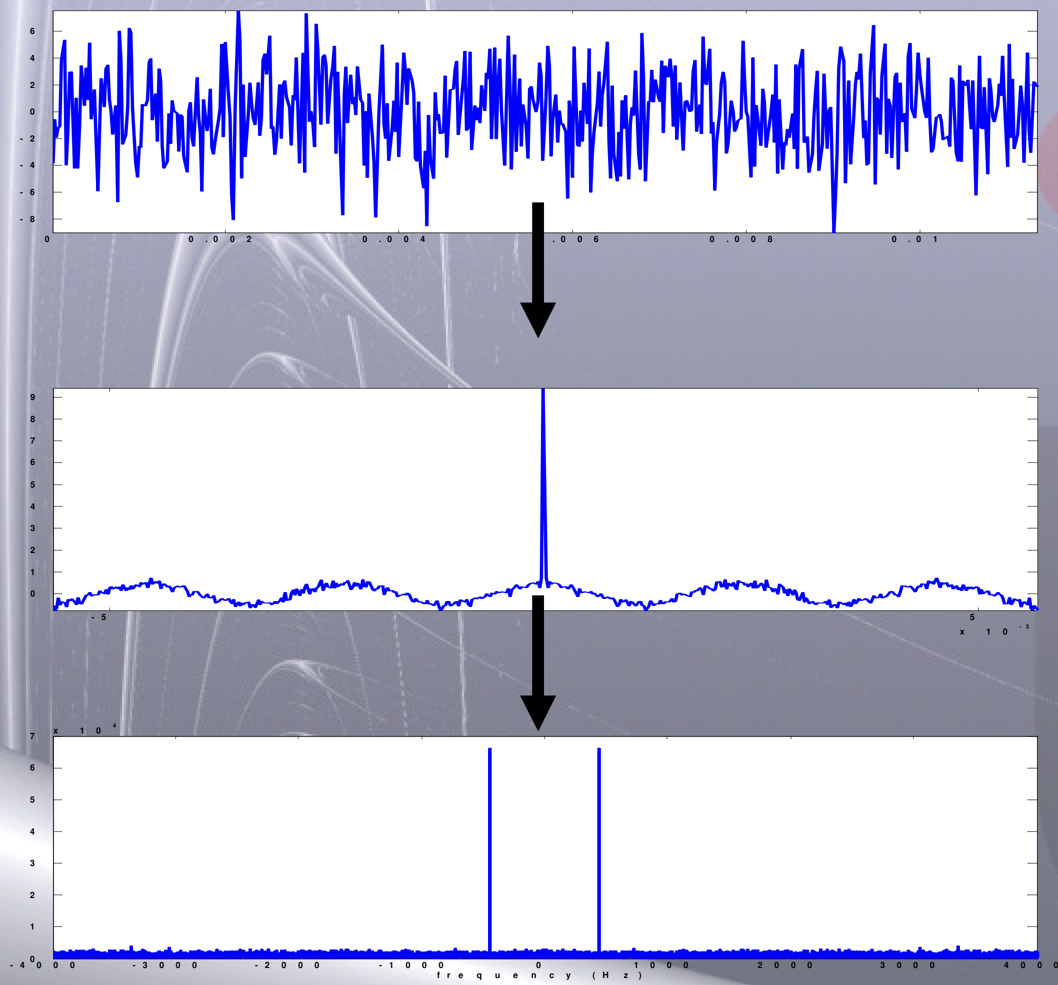
Sine Wave and White Noise



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PSD properties

1. **Symmetry**: PSDs are even functions, $G_x(f)=G_x(-f)$ y $S_x(f)=S_x(-f)$.

2. **Sign**: PSD is real-valued and No-Negative for any value of frequency *f*, $G_x(f) \geq 0$, $S_x(f) \geq 0$.

3. **Integrability:** Energy or Power can be calculated as the integral of the their Spectral Density

Cross Spectral Density

- Let be $x(t)$ and $y(t)$ two energy-defined signals which cross-correlation function is $R_{xy}(\tau)$. Their Cross Spectral Density is defined as the Fourier Transform of their cross-correlation function:

$$G_{xy}(f) = F\{R_{xy}(\tau)\}$$

- Analogously, if $x(t)$ and $y(t)$ are power-defined signals their cross spectral density is defined as:

$$S_{xy}(f) = F\{R_{xy}(\tau)\}$$

Sum of Signals

- Spectral Density of a sum of signals satisfies the following relationship:

$$
\begin{aligned}
S_{\mathbf{x}+\mathrm{y}}(f) &= F\left\{R_{\mathbf{x}+\mathrm{y}}(\tau)\right\} \\
&= F\left\{R_{\mathbf{x}}(\tau) + R_{\mathrm{y}}(\tau) + R_{\mathbf{x}\mathrm{y}}(\tau) + R_{\mathrm{yx}}(\tau)\right\} \\
&= S_{\mathbf{x}}(\tau) + S_{\mathrm{y}}(\tau) + S_{\mathbf{x}\mathrm{y}}(\tau) + S_{\mathrm{yx}}(\tau)
\end{aligned}
$$

Summary of Spectral Density

- Energy (or Power) Spectral Density describes how energy (or power) is distributed along frequencies
- Can be used for Stochastic Processes and deterministic signals
- In the case of SP, Spectral Density represents an statistical average of the process. One particular realization may have different spectrum
- Spectral Density of the sum of several signals can be calculated by computing the Spectral Density of each elementary signal and their cross spectral density

Summary of Concepts in this Chapter

- In this chapter we learned:
  - How to model a Stochastic Process
  - Different properties of SP: Independence, Stationarity, Ergodicity
  - How to compute the autocorrelation of a SP and its physical meaning
  - How to compute the Energy(Power) Spectral Density of a SP and its physical meaning