The Finite Element Method

Section 8

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Section Objectives

In this section we illustrate how the finite element method may be used to tackle problems which vary in the temporal in addition to the spatial domain. Although we will focus on a 1-D implementation, the same techniques are readily extended to problems in additional spatial dimensions.

In this section we will:

- Use the finite element method to approximate a 1-D parabolic PDE: the 1-D Heat Equation
- Illustrate (yet again) the $(S) \rightarrow (W) \rightarrow (G) \rightarrow (M)$ process
- Propose a possible family of solution algorithms

Preliminaries

The example problem which we will use to illustrate this is the parabolic heat equation. This equation is derived from Fourier's Law (which was introduced when we considered the linear heat conduction problem) and the principle of conservation of energy.

We define

$q_i = -\kappa_{ij} u_{,j}$	(Cartesian components of the heat flux vector)
$u(\mathbf{x}, t)$	(temperature)
$\kappa_{ij}(\mathbf{x})$	(conductivity)
ρ	(density)
c	(heat capacity)

Strong Form of the Heat Equation

The strong form (S) of the heat equation is expressed as

$$
(\mathrm{S}) \qquad
\begin{array}{llll}
\rho c u_{,t} + q_{i,i} = l & \text{on} & \Omega\times]0,T[& \\
u = g & \text{on} & \Gamma_g\times]0,T[& \text{(Dirichlet)} \\
-q_i n_i = h & \text{on} & \Gamma_h\times]0,T[& \text{(Neumann)} \\
u(\mathbf{x},0) = u_0(\mathbf{x}) & & \mathbf{x}\in\Omega &
\end{array}
$$

where $u_{,t}$ denotes the derivative of u with respect to time.

As before, the first step is to convert this strong form to a weak integral form.

Weak Form of the Heat Equation

As before $\mathcal{V}$ is the space of weighting functions which do not depend on time.

The space of trial solutions $\mathcal{S}$ is, however, a function of time

$$\mathcal{S} = \left\{ u(\cdot, t) | u(\mathbf{x}, t) = g(\mathbf{x}, t), \ \mathbf{x} \in \Gamma_g, \ u(\cdot, t) \in H^1(\Omega) \right\}$$

Visualizer

Show that the weak form of the heat equation is

$$\text{For} \quad u(\mathbf{x}, t) = g(\mathbf{x}, t), \ \mathbf{x} \in \Gamma_g$$

$$\text{(W)} \qquad (w, \rho c \dot{u}) + a(w, u) = (w, l) + (w, h)_\Gamma$$

in which

$$(w, \rho c u(0)) = (w, \rho c u_0)$$

Galerkin Form of the Heat Equation

We discretize $\mathcal{V}$ to $\mathcal{V}^h$ and $\mathcal{S}$ to $\mathcal{S}^h$. $\mathcal{V}^h$ has individual members v^h i.e. $v^h(t) \in \mathcal{V}^h$. $\mathcal{S}^h$ has individual members u^h which permit the decomposition into $u^h = v^h + g^h$, and consequently $\dot{u}^h = \dot{v}^h + \dot{g}^h$. The temperature boundary conditions are satisfied by $g^h(t)$. More fully

$$u^h(\mathbf{x}, t) = v^h(\mathbf{x}, t) + g^h(\mathbf{x}, t)$$

We are now in a position to obtain the Galerkin form of the heat equation as

$$\text{(G)} \qquad (w^h, \rho c \dot{v}^h) + a(w^h, v^h) = (w^h, l) + (w^h, h)_\Gamma - (w^h, \rho c \dot{g}^h) - a(w^h, g^h)$$

subject to

$$(w^h, \rho c v^h(0)) = (w^h, \rho c u_0) - (w^h, \rho c g^h)$$

This is a semi-discrete equation as time is left continuous.

Shape Functions

We represent g^h and v^h in terms of shape functions as follows

$$v^h(\mathbf{x}, t) = \sum_{A \in \eta - \eta_g} N_A(\mathbf{x}) d_A(t)$$

$$g^h(\mathbf{x}, t) = \sum_{A \in \eta_g} N_A(\mathbf{x}) g_A(t)$$

Note that the shape functions are not time dependent. Time dependence is carried purely by the nodal values. We also have

$$w^h = \sum_{A \in \eta - \eta_g} N_A(\mathbf{x}) c_A$$

Towards the Matrix Form

These approximations are substituted into (G) to give

$$\left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ \rho c \sum_{B\in\eta-\eta_g} N_B \dot{d}_B\right) + a\left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ \sum_{B\in\eta-\eta_g} N_B d_B\right)$$
$$= \left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ l\right) + \left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ h\right)_\Gamma$$
$$-\left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ \rho g \sum_{B\in\eta_g} N_B \dot{g}_B\right) - a\left(\sum_{A\in\eta-\eta_g} N_A c_A\ ,\ \sum_{B\in\eta_g} N_B g_B\right)$$

which may be expressed more concisely as

$$\text{for} \qquad A \in \eta - \eta_g$$

$$\left(N_A\ ,\ \rho c \sum_{B\in\eta-\eta_g} N_B\right) \dot{d}_B + a\left(N_A\ ,\ \sum_{B\in\eta-\eta_g} N_B\right) d_B$$
$$= (N_A,\ ,\ l) + (N_A,\ ,\ h)_\Gamma$$
$$-\left(N_A\ ,\ \rho c \sum_{B\in\eta_g} N_B\right) \dot{g}_B - a\left(N_A\ ,\ \sum_{B\in\eta_g} N_B\right) g_B$$

The Matrix Form

We can then simplify further to give

$$\text{for} \qquad A \in \eta - \eta_g$$

$$\sum_{B \in \eta - \eta_g} (N_A \ , \ \rho c N_B)\dot{d}_b + \sum_{B \in \eta - \eta_g} (N_A \ , \ N_B) d_b$$
$$= (N_A , \ l) + (N_A , \ , \ h)_\Gamma$$
$$\sum_{B \in \eta_g} (N_A \ , \ \rho c N_B)\dot{g}_B - \sum_{B \in \eta_g} a(N_A , \ , N_B) g_B$$

which is simply the matrix equation

$$\text{(M)} \qquad \mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}, \qquad \mathbf{d}(0) = \mathbf{d}_0$$

Assembly

The 'conductivity matrix' (in this case the matrix of heat capacities) M is formed by

$$\mathbf{M} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (\mathbf{m}_e), \qquad \mathbf{m}_e = [m^e_{ab}], \qquad m^e_{ab} = \int_{\Omega^e} N_a \rho c N_b d\Omega$$

The 'stiffness' matrix, as before, is

$$\mathbf{K} = \mathop{\mathbf{A}}_{e=1}^{n_{el}} (\mathbf{k}_e), \qquad \mathbf{k}_e = [k^e_{ab}], \qquad k^e_{ab} = \int_{\Omega^e} \mathbf{B}_a^T \mathbf{D} \mathbf{B}_b d\Omega$$

The force vector is

$$\mathbf{F}(t) = \mathbf{F}_{nodal}(t) + \mathop{\mathbf{A}}_{e=1}^{n_{el}} (\mathbf{f}^e(t))$$

where

$$\mathbf{f}_e = \{f^e_a\}, \qquad f^e_a = \int_{\Omega^e} N_a l d\Omega + \int_{\Gamma^h_h} N_a h d\Gamma - \sum_{b=1}^{n_{en}} (k^e_{ab} g^e_b + m^e_{ab} \dot{g}^e_b)$$

Lumped **M** Matrix

Note that the symmetry and bilinearity of its component functions means that **M** is symmetric and positive definite. Also, importantly, the definition is variationally consistent.

This means that its definition has been derived from first principles starting from the weak form of the initial equation. Such a definition leads to optimal error estimates in the finite element solution.

However, in some cases (such as in *explicit* solution algorithms) it can be beneficial to define the ‘mass’ matrix differently such that it is diagonal i.e. all its terms are ‘lumped’ onto the diagonal. This diagonalization can be achieved via the application of nodal quadrature (see tutorial problem set).

Algorithms for the Solution of Parabolic Problems

Recall the semi-discrete matrix form of a parabolic problem is

$$\mathbf{M}\dot{\mathbf{d}} + \mathbf{K}\mathbf{d} = \mathbf{F}$$

in which

$\mathbf{M}$	Capacity matrix
$\mathbf{K}$	Conductivity matrix
$\mathbf{F}$	Heat Supply vector
$\mathbf{d}$	Temperature vector

and the heat supply vector is itself a function of time, i.e.

$$\mathbf{F} = \mathbf{F}(t)$$

We want to solve the initial value problem: find $\mathbf{d}(t)$ which satisfies the above equation subject to $\mathbf{d}(0) = \mathbf{d}_0$.

Visualizer

Demonstrate the Trapezoidal Family of Solution Schemes

Summary

We have now extended the finite element method to account for time-varying problems. The techniques used to solve the 1-D Parabolic Heat Equation can readily be extended for problems in 2 or 3 spatial dimensions by following a similar approach.

Particular points are that:

- by placing time variation onto the nodal values the assembly operator, shape functions, and $\mathbf{K}$ matrix are inherited from the classical linear heat conduction approximation
- symmetric bilinear weak forms are *very* convenient
- time marching methods may be used to evaluate the time-dependent solution

In conclusion, the finite element method is a very powerful technique.