The Finite Element Method

Section 10

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Section Objectives

We now have a firm mathematical grounding of the principles behind the finite element method. However, it is of interest to go back to the original *Direct Stiffness Method* which was developed in the 1930s from physical principles. This provides a starting point for a more 'structural' approach to finite element analysis but several parallels with the mathematical approach will become clear.

In this section we will:

- Contextualize the motivation behind the development of matrix methods for structural analysis
- Introduce the stiffness matrix for a simple rod element
- Use the direct stiffness method to solve for forces and displacements in a statically-indeterminate framework.

Background

In the 1930s, aircraft structures were becoming increasingly complicated and less amenable to solution by traditional hand calculation techniques.

The Mk. 2 Vickers Wellington: geodetic sub-structure

Background

The geodetic skeleton of the Wellington bomber allowed it to sustain an incredible amount of battle damage.

Battle damage in the tail of a Wellington Bomber

The analysis complexity of such a structure is, however, greatly increased. By noting that the structure consists of a large number of similar connected sub-components we start to move towards a finite element-type analysis methodology.

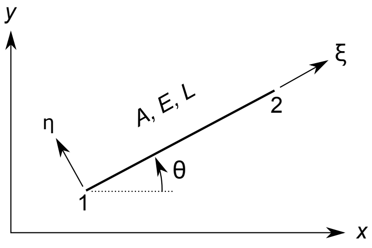
Direct Stiffness Method

Our approach is to characterize the force-displacement relationship for an individual sub component using classical structural analysis, which allows us to determine its stiffness. By coupling the interconnected stiffnesses we can assemble a matrix which represents the stiffness of the complete structure.

By way of illustration we will consider one of (if not) the simplest structural components: the unidirectional rod. This can sustain tensile or compressive forces in the axial direction, and will extend or compress in length accordingly in response.

The approach can be extended and refined for more complex structural components — beams, beam columns, plates, shells — until eventually becoming practically indistinguishable from the finite element method.

The Uniaxial Structural Rod Element



We introduce a uniaxial rod of length L, cross-sectional area A, fabricated from a linear elastic material of Young's modulus E. The rod is defined with respect to local axial coordinate ξ and transverse coordinate η, and is positioned an an orientation θ with respect to the global coordinate system (x, y).

At ends (nodes) 1 and 2 we define axial forces p_1 and p_2, and corresponding displacements u_1 and u_2 all defined in the positive direction ξ.

Directly Calculated Elemental Stiffness Matrix

From force equilibrium we can immediately write that $p_1 = -p_2$.

We then apply Hooke's Law in 1-D to determine the relationship between force and displacement at each node in local coordinates

$$\begin{aligned} \sigma &= E\epsilon \quad \text{where } \sigma \text{ is stress, and } \epsilon \text{ is strain} \\ \frac{p_1}{A} &= \frac{E(u_1 - u_2)}{L} \end{aligned}$$

hence

$$p_2 = -p_1 = \frac{AE}{L}(u_2 - u_1)$$

which in matrix form gives

$$\left\{ \begin{array}{c} p_1 \\ p_2 \end{array} \right\} = \frac{AE}{L} \left[\begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right] \left\{ \begin{array}{c} u_1 \\ u_2 \end{array} \right\}$$

or

$$\mathbf{p} = k\mathbf{u}$$

Global-to-Local Coordinate Transformation

We can relate global displacements **U** to local displacements **u** via the direction cosines

$$
\begin{aligned}
u &= U\cos\theta + V\sin\theta \\
v &= -U\sin\theta + V\cos\theta
\end{aligned}
$$

or more concisely for the displacements at both ends as

$$
\left\{\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}\right\} = \begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & 0 & -s & c \end{bmatrix} \left\{\begin{array}{c} U_1 \\ V_1 \\ U_2 \\ V_2 \end{array}\right\}
$$

Local-to-Global Coordinate Transformation

If we define $\mathbf{P}$ as the vector of horizontal and vertical externally-applied nodal forces we can relate them to the local axial forces by the same transformation. This gives

$$\begin{aligned}\mathbf{u} &= D\mathbf{U} \\ \mathbf{p} &= D\mathbf{P}\end{aligned}$$

where D is the matrix of direction cosines. Matrix D can readily be shown to be orthogonal and hence $D^{-1} = D^\top$.

We can therefore express the local-to-global tranformation as

$$\begin{aligned}\mathbf{U} &= D^\top\mathbf{u} \\ \mathbf{P} &= D^\top\mathbf{p}\end{aligned}$$

Global Elemental Stiffness Matrix

We can generate a global elemental stiffness matrix as follows. We have already defined

$$\mathbf{p} = k\mathbf{u}$$

Applying the local-to-global transformation gives

$$\mathbf{P} = [D^\top kD]\mathbf{U} \quad \text{or} \quad \mathbf{P} = K\mathbf{U}$$

local stiffness matrix

$$k = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

global stiffness matrix

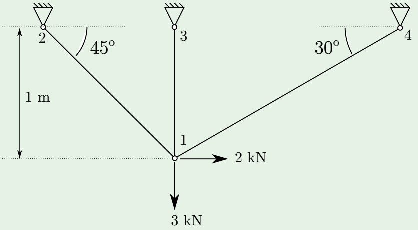
$$K = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix}$$

Assembly and Solution

Having defined an elemental stiffness matrix for the 1-D structural rod element in global coordinates, the analysis of frameworks consisting of several rod sub-components is best illustrated by example.

Visualizer

Find the bar forces and deflected shape for the statically-indeterminate framework shown using the direct stiffness method.



Summary

We have very briefly introduced the direct stiffness method — a direct antecedent of the finite element method.

Although it is much less refined, many of the same concepts are involved, including:

- local element definition
- local-to-global transformation
- assembly of full system stiffness matrices