CEU
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UNIT 2 – Part III: Discrete Random Processes

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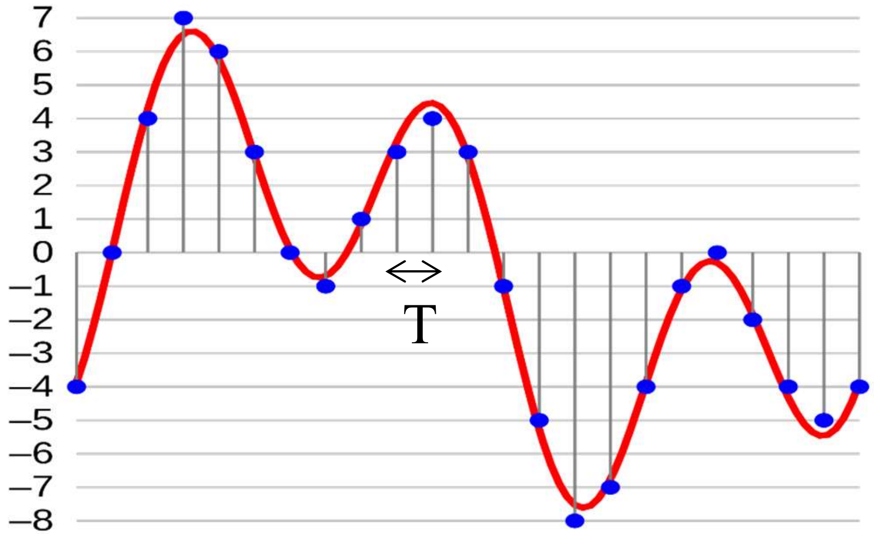
Discrete signals

- A discrete signal *x[n]* results from the sampling of a continuous signal $x_c(t)$ with a sampling period T ($\mathrm{f_s} = \frac{1}{\mathrm{T}}$)

$$x[n] = x_{\mathrm{c}}(nT)$$

where *n* is an integer number



- The sampling frequency must be higher than twice bandwidth of the continuous signal (**Nyquist theorem**)

- x[n] can now be stored in a digital system (after quantizing its amplitude as a binary number) and it can be processed by means of **digital signal processing** techniques

Fourier transform

- We can define now the equivalent to the **Laplace transform** for discrete signals: the **Z transform**

$$X(Z) = \sum_{n=-\infty}^{\infty} x[n] Z^{-n}$$

- The **Fourier transform** of discrete signals is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega}$$

- Note that the FT is periodic with period 2π
- The Interval from 0 to 2π is equivalent to frequencies from $-\frac{f_s}{2}$ (- π rad/s) to $\frac{f_s}{2}$ ($+\pi$ rad/s)
- The ZT and the FT are instrumental in the analysis of LTI systems
- In discrete systems, the delta function (Kronecker's delta) is defined as

$$\delta[n] = \begin{cases} 1, & if\ n = 0 \\ 0, & otherwise \end{cases}$$

$\delta[0]$ is not infinite!

Digital filters

- A digital filter is characterized through a discrete impulse response *h[n]*
- The output *y[n]* of a filter with impulse response *h[n]* and input *x[n]* is

$$\mathrm{y[n]} = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- It is posible to rewrite y[n] and relate it to the frequency response

$$\mathrm{y[n]} = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

$$\mathrm{H(Z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}$$

Digital filters

- Finite impulse response (FIR) filter are those whose output only depends on the input (i.e. a_k=0)
- The impulse response is finite in time

$$\mathrm{y[n]} = \sum_{k=0}^{M} b_k x[n-k]$$

$$\mathrm{H(Z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$\mathrm{h[n]} = \sum_{k=0}^{M} b_k \delta[n-k]$$

Digital filters

- For infinite impulse response (IIR) filters, the output is a function of the input and past outputs (i.e. $a_k \neq 0$)
- The impulse response is infinite in time, since it depends on the last N outputs

$$\mathrm{y[n]} = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k]$$

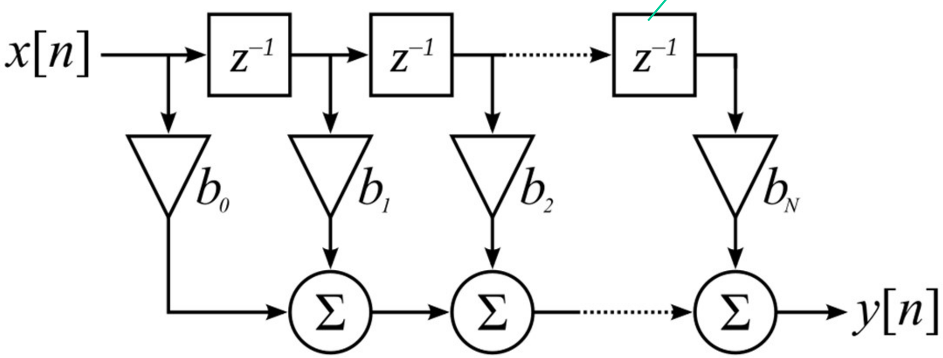
$$\mathrm{H(Z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1-\sum_{k=1}^{N} a_k z^{-k}}$$

$$\mathrm{h[n]} = \sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k \delta[n-k]$$

Digital filters

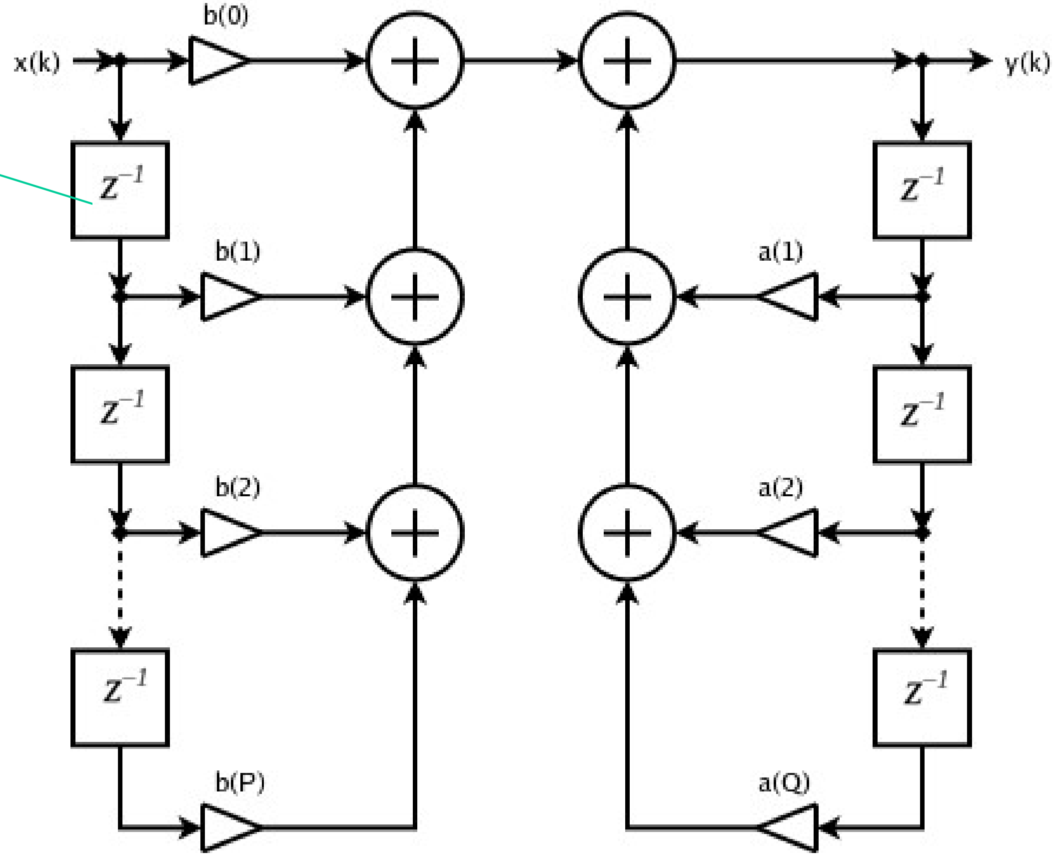
- Example of filter structures

FIR



1 –sample delay

IIR



Discrete random processes

- Given two w.s.s. discrete random process $X[n]$ and $Y[n] = X[n] * h[n]$

$$\mathrm{E}\big[\mathrm{X}[\mathrm{n}]\big] = \bar{X}$$

$$\mathrm{E}\big[Y[\mathrm{n}]\big] = \sum_{k=-\infty}^{\infty} h[k]E[X[n-k]] =$$

$$\bar{X}\sum_{k=-\infty}^{\infty} h[k] = \bar{X}H(e^{j0})$$

Discrete random processes

$$\mathrm{R_{YY}}[\tau] = \mathrm{E}[\mathrm{Y}[\mathrm{n}]\mathrm{Y}[\mathrm{n}+\tau]] = \sum_{k=-\infty}^{\infty} h[k] \sum_{r=-\infty}^{\infty} h[r] R_{XX}[\tau + k - r]$$

$$\mathrm{S_{YY}}(\omega) = \left|H\left(e^{j\omega}\right)\right|^2 S_{XX}(\omega)$$

Note the new integral limits

$$P_{\mathrm{YY}} = R_{YY}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{YY}(\omega)d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right)\right|^2 S_{XX}(\omega)d\omega$$

Discrete random processes

- Given If X[n] is a white noise, then

$$R_{XX}[\tau] = \frac{N_0}{2}\delta[n]$$

$$S_{XX}(\omega) = \frac{N_0}{2}$$

$$P_{XX} = R_{XX}[0] = \frac{N_0}{2}$$

Note that the power of a discrete White noise is **not** infinite

$$R_{YY}[\tau] = \frac{N_0}{2}\sum_{k=-\infty}^{\infty} h[k]h[k+\tau]$$

$$S_{YY}(\omega) = \left|H\left(e^{j\omega}\right)\right|^2 S_{XX}(\omega) = \left|H\left(e^{j\omega}\right)\right|^2 \frac{N_0}{2}$$

$$P_{YY} = R_{YY}[0] = R_{XX}[0]\sum_{k=-\infty}^{\infty} h[k]^2 =$$

$$P_{XX}\sum_{k=-\infty}^{\infty} h[k]^2 =$$

$$\frac{N_0}{2}\sum_{k=-\infty}^{\infty} h[k]^2 =$$

$$\left(\frac{N_0}{2}\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\omega}\right)\right|^2 d\omega\right)$$

Parseval's theorem

Discrete random processes

- The time averages can be computed as

$$\mathrm{A}\left[\mathrm{x}[\mathrm{n}]\right] = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{k=-L}^{L} x[k]$$

$$\mathbb{R}\_xx\left[\mathrm{x}[\mathrm{n}]x[n+\tau]\right] = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{k=-L}^{L} x[k]\, x[k+\tau]$$

SUMMARY

- Discrete signals
- Digital filters
- Discrete random processes
  - Mean, autocorrelation, power spectrum
  - LTI systems
  - Time averages