CEU
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UNIT 2 – Part I: Random Processes
Temporal Characteristics

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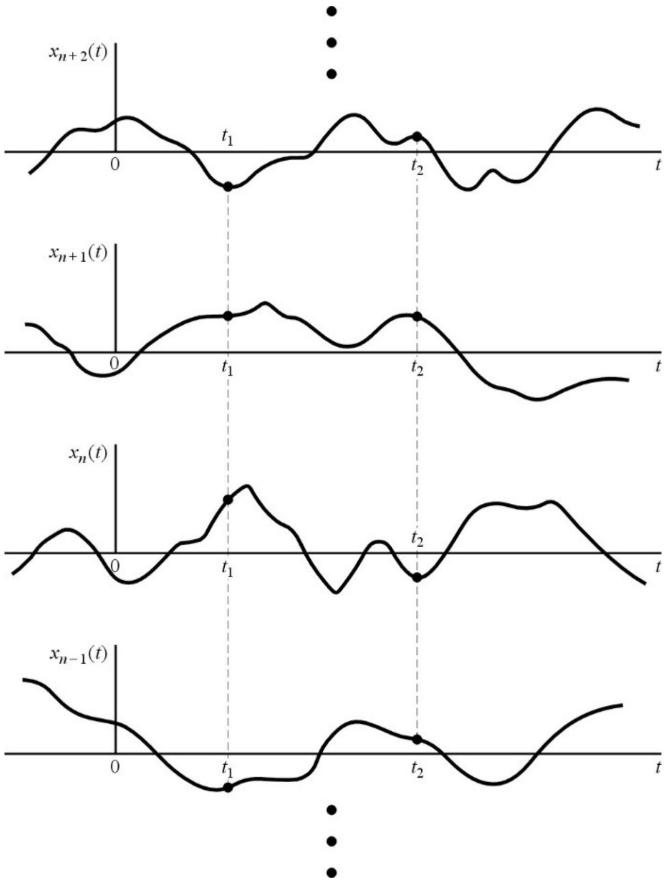
Random Processes

Given a function x*(t,s)* that relates the elements in the sample space *s* and time *t*

- A random process *X(t,s)* is a family or *ensemble* of *x(t,s)* functions
- Each function is called a sample or ensemble function
- $X_i=X(t_i,s)$ is a random variable (time is fixed to t_i)
- *The statistics of* X_i are the statistics of the process

  at time $t=t_i$



EXAMPLE

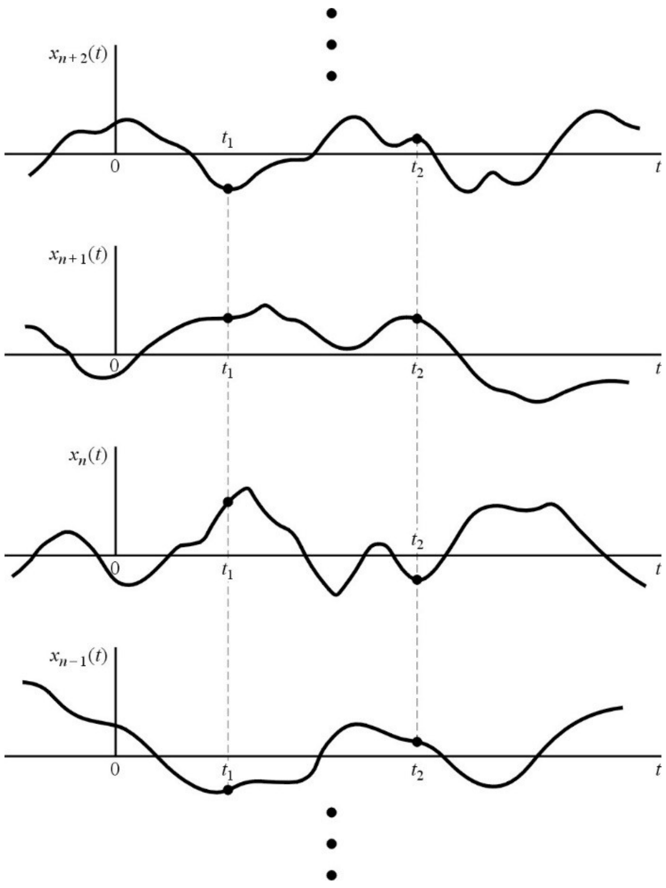
$$X(t) = a\cos(\omega_0 t + \varphi),$$

$\varphi \sim U(0,2\pi)$

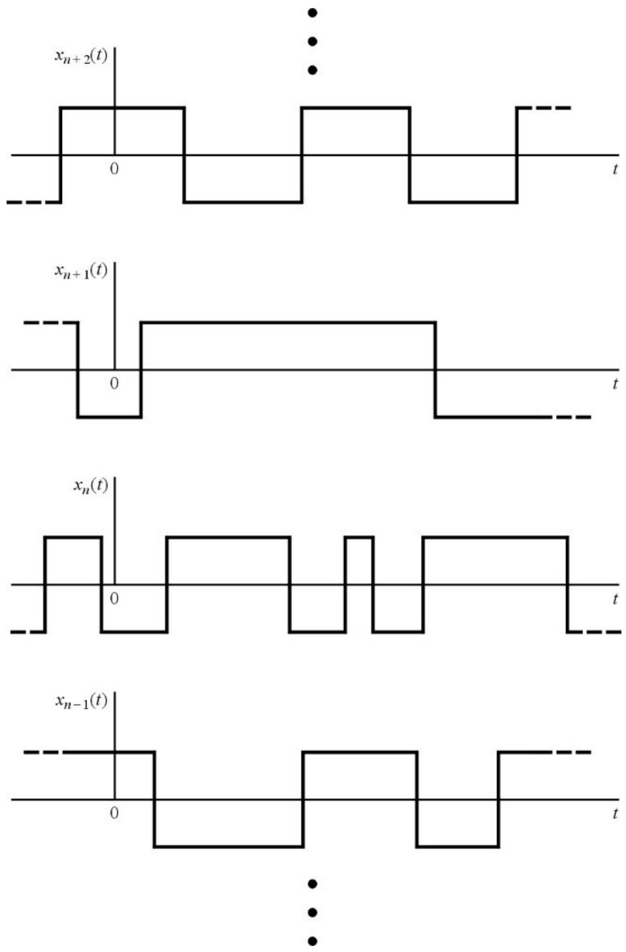
Random Processes

Classification:

Continuous



Discrete



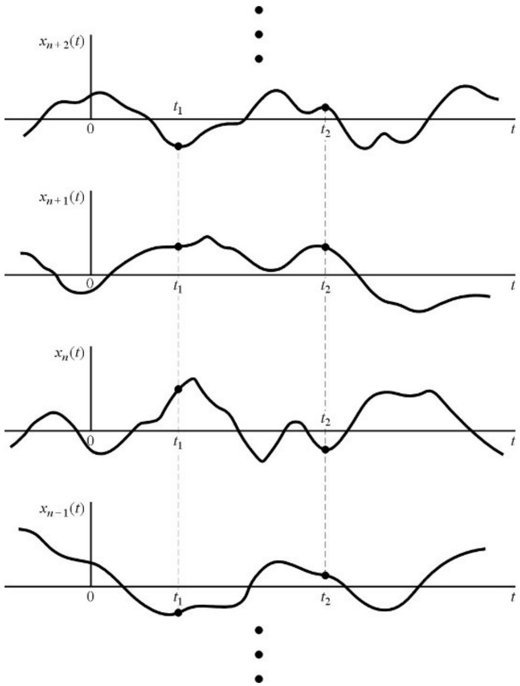
Random Processes

Classification:

- Deterministic: Future values of the process can be predicted from past values

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0,2\pi)$$

- Undeterministic: It is not possible to predict future values



Distribution Function

- For a given time t_1 the distribution function is defined as

$$F_X(x_1;t_1) = P\{X(t_1) \leq x_1\}$$

- For two random variables $X_1=X(t_1)$ and $X_2=X(t_2)$

$$F_X(x_1,x_2;t_1,t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\}$$

- And for *n* random variables

$$F_X(x_1,\ldots,x_N;t_1,\ldots,t_N) = P\{X(t_1) \leq x_1,\ldots,X(t_N) \leq x_N\}$$

Density Function

- The probability density functions for one, two and *n* r.v. are

$$f_X(x_1;t_1) = \frac{dF_X(x_1;t_1)}{dx}$$

$$f_X(x_1,x_2;t_1,t_2) = \frac{\partial^2 F_X(x_1,x_2;t_1,t_2)}{\partial x_1 \partial x_2}$$

$$f_X(x_1,\ldots,x_N;t_1,\ldots,t_N) = \frac{\partial^N F_X(x_1,\ldots,x_N;t_1,\ldots,t_N)}{\partial x_1 \ldots \partial x_N}$$

Independency

- Two processes *X* and *Y* are independent if

$$f_{XY}(x_1,\dots,x_N,y_1,\dots,y_N;t_1,\dots,t_N,t'_1,\dots,t'_N)=$$
$$f_X(x_1,\dots,x_N;t_1,\dots,t_N)f_Y(y_1,\dots,y_N;t'_1,\dots,t'_N)$$

First-order stationary processes

- A random process is *stationary to order one* if the p.d.f. does not change with a shift in time origin

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta)$$

- This implies that

$$E[X(t_1)] = \overline{X} = \text{constant}$$

Second-order stationarity

- A random process is *stationary to order two* if the p.d.f. satisfies

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta)$$

- This implies that $R_{XX}(t_1,t_2)$, called ***autocorrelation***, is a function of $\tau = t_2 - t_1$

$$R_{XX}(t_1, t_2) = R_{XX}(t_1, t_1 + \tau) = E[X(t_1)X(t_1 + \tau)] = R_{XX}(\tau)$$

The correlation applied to $X(t_1)$ and $X(t_2)$ is called the **autocorrelation**

Second-order stationarity

- EXAMPLE: Show that $X(t) = A\cos(\omega_0 t + \Theta)$ is stationary to order two.

  A and ω_0 are constants and $\Theta \sim U(0,2\pi)$

$$R_{XX}(t_1, t_1 + \tau) = \frac{A^2}{2}\cos(\omega_0 \tau) = R_{XX}(\tau)$$

Second-order stationarity

- EXAMPLE: Check if $X(t) = A\cos(\omega_0 t + \Theta)$ is first- and second-order stationary.

  A and ω_0 are constants and $\Theta \sim \mathrm{U}(0, \pi)$

$$\mathrm{E}[\mathrm{X}(\mathrm{t})] = \frac{\mathrm{A}}{2\pi}\sin(\omega_0 t)$$

$$R_{XX}(t_1, t_1 + \tau) = \frac{A^2}{2}\cos(\omega_0 \tau) = R_{XX}(\tau)$$

Wide-sense and strict-sense stationarity

- If a process *X* is stationary to orders <u>one</u> and <u>two</u> it is said to be **wide-sense stationary** (w.s.s.)

$$E[X(t_1)] = \text{constant}$$

$$R_{XX}(t_1, t_2) = R_{XX}(\tau)$$

- Two w.s.s. processes *X* and *Y* are jointly wide-sense stationary if they are w.s.s. and

$$R_{XY}(t_1, t_2) = R_{XY}(\tau)$$

Now, it is called the **cross-correlation**

- A random process is ***strict-sense stationary*** if it is stationary to any order *N*

$$f_X(x_1, \ldots, x_N; t_1, \ldots, t_N) = f_X(x_1, \ldots, x_N; t_1 + \Delta, \ldots, t_N + \Delta)$$

Wide-sense and strict-sense stationarity

- **CHALLENGE**: Show graphically that $X(t) = A\cos(\omega_0 t + \varphi)$ is not stationary.

  ω_0 and ϕ are constants and $A \sim U(0,1)$

Ergodicity

- The time average of a quantity is defined as

$$A[\cdot] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} [\cdot] dt$$

- The **time average** of a sample function **x(t)** is

$$\overline{x} = A[x(t)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t) dt$$

- The **time autocorrelation** function is

$$\Re xx(\tau) = A[x(t)x(t+\tau)] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t+\tau) dt$$

Ergodicity

- A wide-sense stationary process X is **ergodic in the mean** if

$$E[X] = \overline{X} = A[x(t)] = \overline{x}$$

- A wide-sense stationary process X is **ergodic in the autocorrelation** if

$$R_{XX}(\tau) = \Re xx(\tau)$$

Computing the time averages of a **single** sample function gives us the statistics of the process

- Two jointly wide-sense processes X and Y are **jointly ergodic** if they are individually ergodic and

$$R_{XY}(\tau) = \Re xy(\tau)$$

Correlation, cross-correlation and covariance

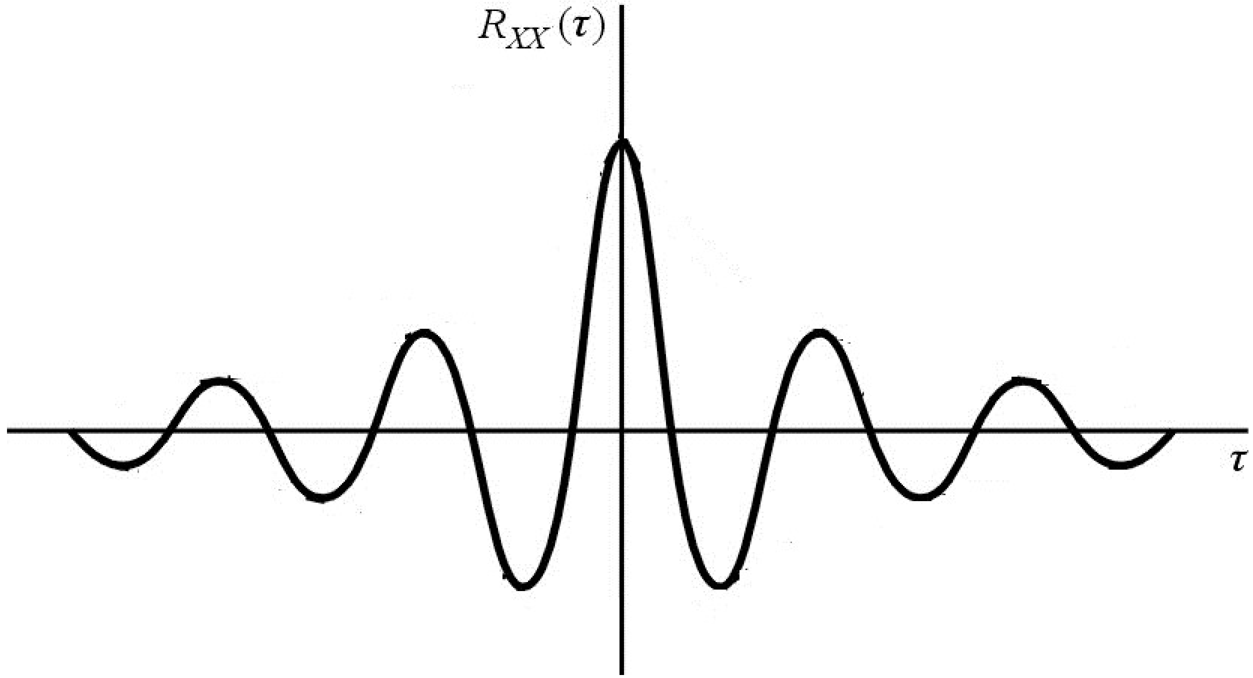
- Some properties of the **autocorrelation** of a **w.s.s.** process

1. $|R_{XX}(\tau)| \leq R_{XX}(0)$

2. $R_{XX}(-\tau) = R_{XX}(\tau)$

3. $R_{XX}(0) = E[X^2(t)]$

Power of the process



Correlation, cross-correlation and covariance

- Some properties of the **cross-correlation** of **w.s.s.** processes

  1. $R_{XY}(-\tau) = R_{YX}(\tau)$

  2. $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)} \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$

- Two processes are **orthogonal** if

$$R_{XY}(t, t+\tau) = 0$$

- Two processes are **uncorrelated** if

$$R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)] \xrightarrow{w.s.s.} = \overline{X}\,\overline{Y}$$

Correlation, cross-correlation and covariance

- EXAMPLE: Given two w.s.s. random processes

$$X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$
$$Y(t) = B\cos(\omega_0 t) - A\sin(\omega_0 t)$$

where *A* and *B* are uncorrelated, zero-mean r.v. with the same variance, check if X and Y are jointly wide-sense stationary

$$R_{XY}(t_1, t_1+\tau) = -\sigma^2 \sin(\omega_0 \tau) = R_{XY}(\tau)$$

Correlation, cross-correlation and covariance

- The **autocovariance** of X is

$$C_{XX}(t,t+\tau) = E\big[\big(X(t)-E[X(t)]\big)\big(X(t+\tau)-E[X(t+\tau)]\big)\big]$$
$$= R_{XX}(t,t+\tau) - E[X(t)]E[X(t+\tau)]$$

If X is at least w.s.s. $$C_{XX}(\tau) = R_{XX}(\tau) - (\overline{X})^2$$

- The **cross-covariance** of X and Y is

$$C_{XY}(t,t+\tau) = E\big[\big(X(t)-E[X(t)]\big)\big(Y(t+\tau)-E[Y(t+\tau)]\big)\big]$$
$$= R_{XY}(t,t+\tau) - E[X(t)]E[Y(t+\tau)]$$

If X and Y are at least **jointly w.s.s.**

$$C_{XY}(\tau) = R_{XY}(\tau) - \overline{X}\,\overline{Y}$$

Correlation, cross-correlation and covariance

- The **variance** of X can be computed from the **autocovariance**

$$\sigma_X^2 = E\left[\left(X(t) - E[X(t)]\right)^2\right] = C_{XX}(t,t) \xrightarrow{w.s.s.} C_{XX}(0) = R_{XX}(0) - \overline{X}^2$$

- Two processes X and Y are **uncorrelated** if

$$C_{XY}(t, t+\tau) = 0$$

$$R_{XY}(t, t+\tau) = E[X(t)]E[Y(t+\tau)]$$

- Remember that **independency** implies **uncorrelation** but not the other way around

SUMMARY

- Random processes
- Distribution and density functions
- Stationarity and ergodicity
- Auto- and Cross-Correlation
- Auto- and Cross-covariance
- Orthogonality
- Independence
- Correlation