

Control y Automatización, IOI

Modelado matemático. Transformada de Laplace. Función de transferencia

transformada de Laplace

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$$

dos perspectivas del
mismo fenómeno

$$f(t) \rightarrow F(s)$$

vista de la señal
en tiempo

vista de la señal
en frecuencia

convierte ecuaciones diferenciales en ecuaciones algebraicas

relación algebraica entre la entrada y la salida del sistema: **Función de transferencia**

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

TABLE 2.1 Laplace transform table

Item no.	$f(t)$	$F(s)$
1.	$\delta(t)$	1
2.	$u(t)$	$\frac{1}{s}$
3.	$tu(t)$	$\frac{1}{s^2}$
4.	$t^n u(t)$	$\frac{n!}{s^{n+1}}$
5.	$e^{-at} u(t)$	$\frac{1}{s+a}$
6.	$\sin \omega t u(t)$	$\frac{\omega}{s^2 + \omega^2}$
7.	$\cos \omega t u(t)$	$\frac{s}{s^2 + \omega^2}$

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TABLE 2.2 Laplace transform theorems

Item no.	Theorem	Name
1.	$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} dt$	Definition
2.	$\mathcal{L}[kf(t)] = kF(s)$	Linearity theorem
3.	$\mathcal{L}[f_1(t) + f_2(t)] = F_1(s) + F_2(s)$	Linearity theorem
4.	$\mathcal{L}[e^{-at}f(t)] = F(s + a)$	Frequency shift theorem
5.	$\mathcal{L}[f(t - T)] = e^{-sT}F(s)$	Time shift theorem
6.	$\mathcal{L}[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right)$	Scaling theorem
7.	$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$	Differentiation theorem
8.	$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - sf(0-) - f'(0-)$	Differentiation theorem
9.	$\mathcal{L}\left[\frac{d^nf}{dt^n}\right] = s^nF(s) - \sum_{k=1}^n s^{n-k}f^{k-1}(0-)$	Differentiation theorem
10.	$\mathcal{L}\left[\int_{0-}^t f(\tau)d\tau\right] = \frac{F(s)}{s}$	Integration theorem
11.	$f(\infty) = \lim_{s \rightarrow 0} sF(s)$	Final value theorem ¹
12.	$f(0+) = \lim_{s \rightarrow \infty} sF(s)$	Initial value theorem ²

¹For this theorem to yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

²For this theorem to be valid, $f(t)$ must be continuous or have a step discontinuity at $t = 0$ (that is, no impulses or their derivatives at $t = 0$).

Table 2.2

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Transformada de Laplace de una función

$$f(t) \rightarrow F(s)$$

Transformada inversa de Laplace de una función

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t)$$

expansión en fracciones parciales:

raíces reales distintas

raíces reales múltiples

raíces complejas o imaginarias puras

Función de transferencia a partir de una ecuación diferencial

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_0 r(t)$$

$$a_n s^n C(s) + a_{n-1} s^{n-1} C(s) + \dots + a_0 C(s) + [\text{cond. iniciales } c(t)] = b_m s^m R(s) + b_{m-1} s^{m-1} R(s) + b_0 + [\text{cond. iniciales } r(t)]$$

Función de
transferencia

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0}$$

Respuesta del sistema a partir de la función de transferencia, $G(s)$, mediante T.L inversa

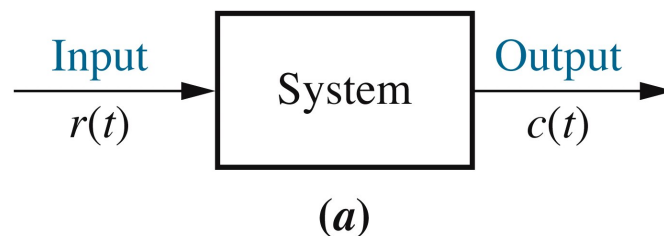


Figure 2.1a
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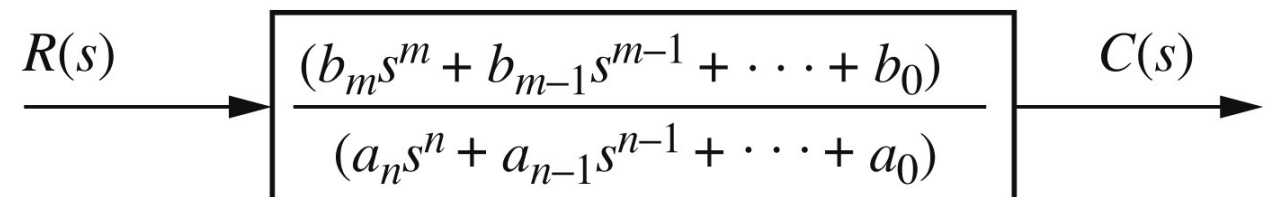


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$$C(s) = G(s) \cdot R(s)$$

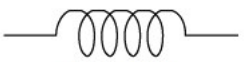
$$C(s) \xrightarrow{\mathcal{L}^{-1}} c(t)$$

descomponemos $C(s)$ en fracciones simples, de las cuales conocemos su transformada inversa de Laplace, y así obtenemos $c(t)$

Concepto de impedancia

Impedancia en circuitos eléctricos con R, L, C

TABLE 2.3 Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C} \int_0^t i(\tau) d\tau$	$i(t) = C \frac{dv(t)}{dt}$	$v(t) = \frac{1}{C} q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R} v(t)$	$v(t) = R \frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L \frac{di(t)}{dt}$	$i(t) = \frac{1}{L} \int_0^t v(\tau) d\tau$	$v(t) = L \frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Note: The following set of symbols and units is used throughout this book: $v(t)$ – V (volts), $i(t)$ – A (amps), $q(t)$ – Q (coulombs), C – F (farads), R – Ω (ohms), G – Ω (mhos), L – H (henries).

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Circuito RLC:

solución mediante ecuación diferencial // solución mediante T.L, impedancia

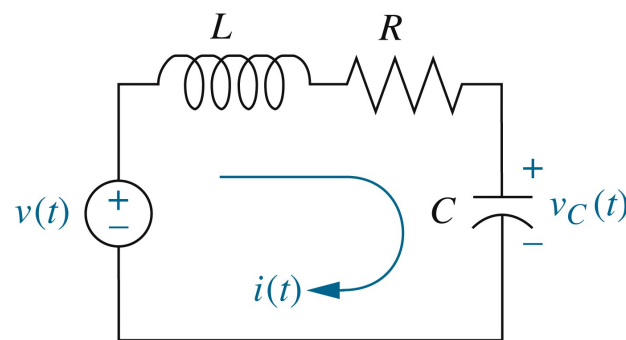


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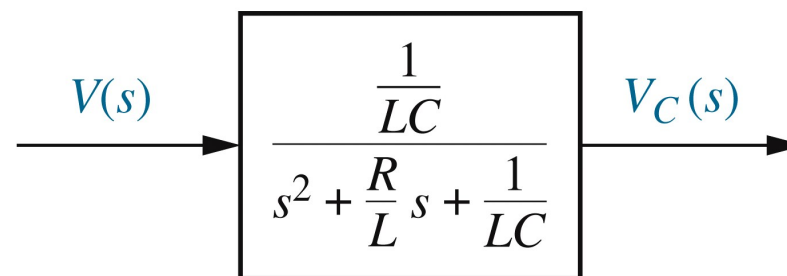


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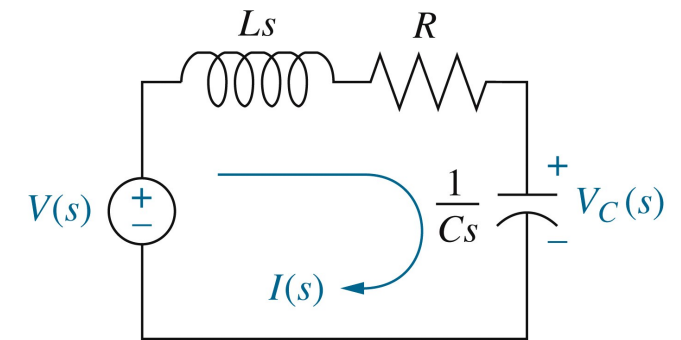
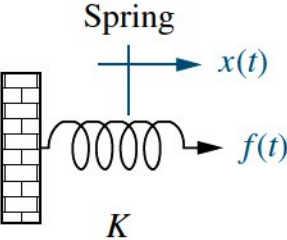
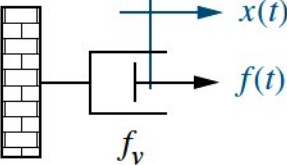
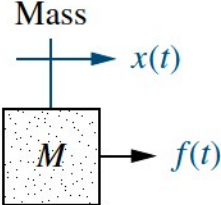


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Sistemas mecánicos de traslación

TABLE 2.4 Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	Force-velocity	Force-displacement	Impedance $Z_M(s) = F(s)/X(s)$
Spring 	$f(t) = K \int_0^t v(\tau) d\tau$	$f(t) = Kx(t)$	K
Viscous damper 	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M \frac{dv(t)}{dt}$	$f(t) = M \frac{d^2x(t)}{dt^2}$	Ms^2

Note: The following set of symbols and units is used throughout this book: $f(t)$ = N (newtons), $x(t)$ = m (meters), $v(t)$ = m/s (meters/second), K = N/m (newtons/meter), f_v = N-s/m (newton-seconds/meter), M = kg (kilograms = newton-seconds²/meter).

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Diagramas de bloque

Cascada

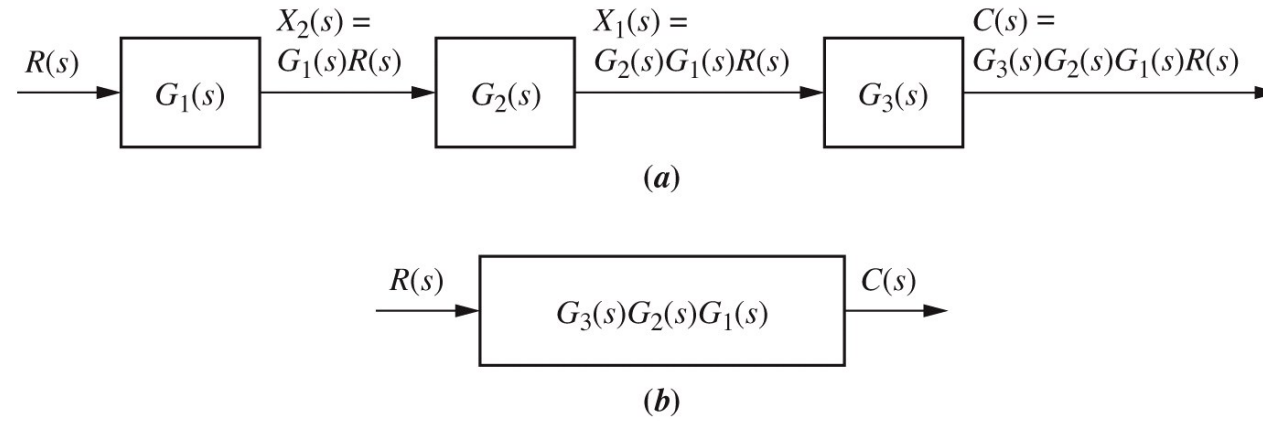


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Paralelo

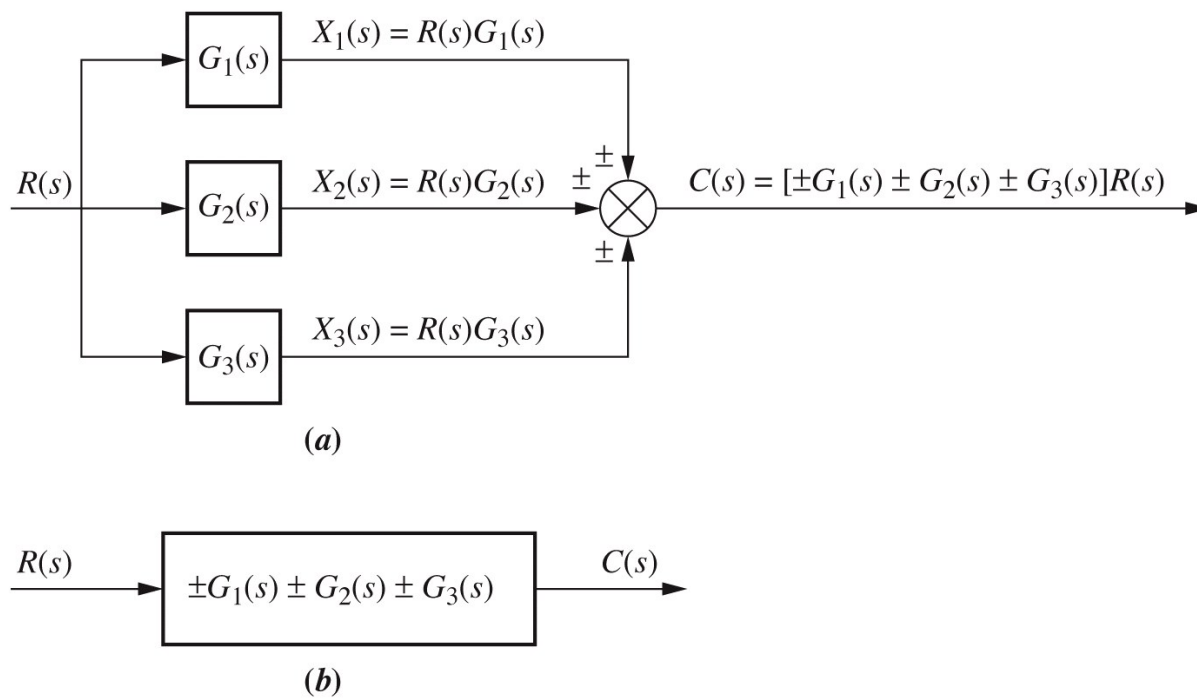


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Feedback

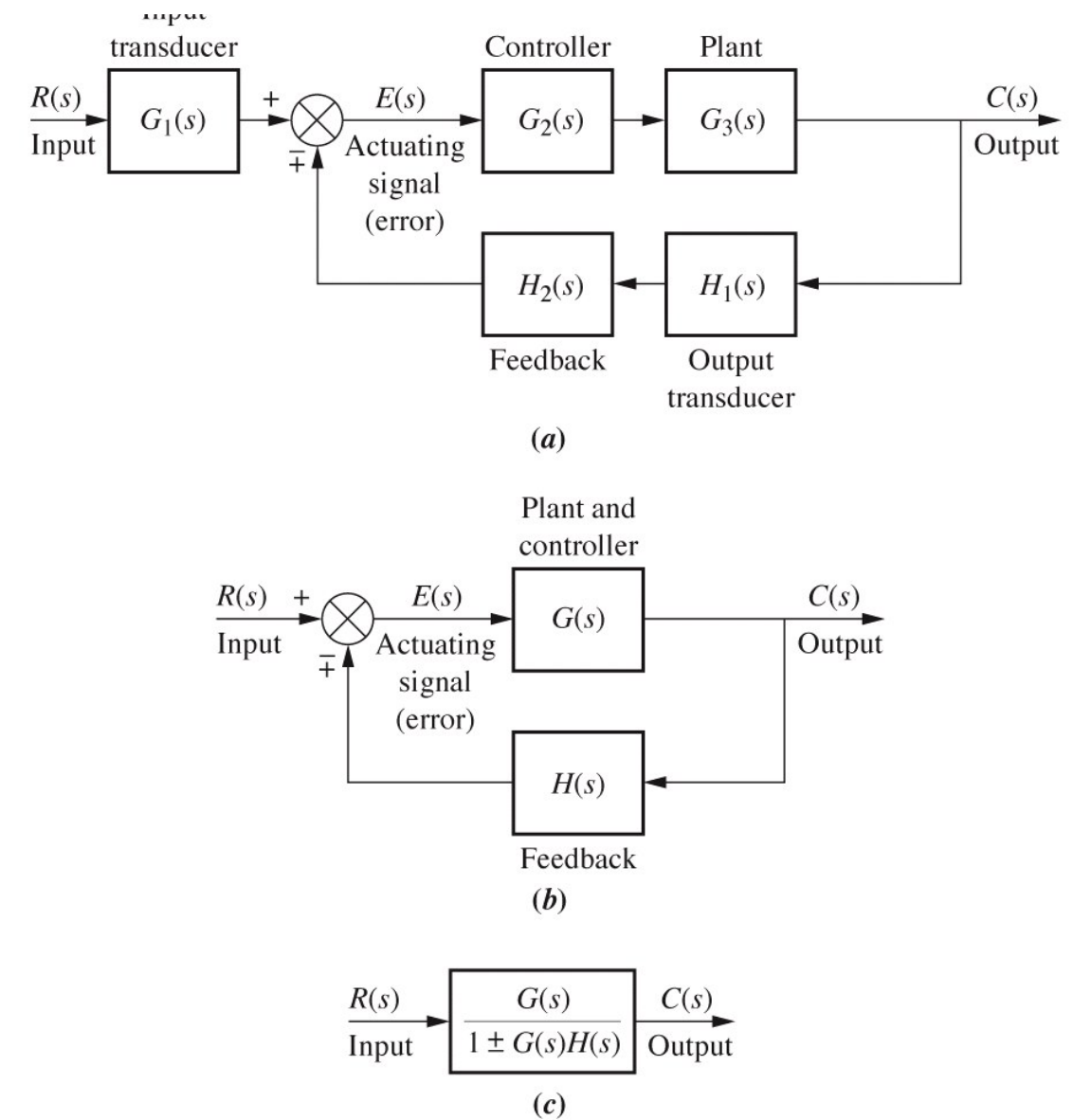
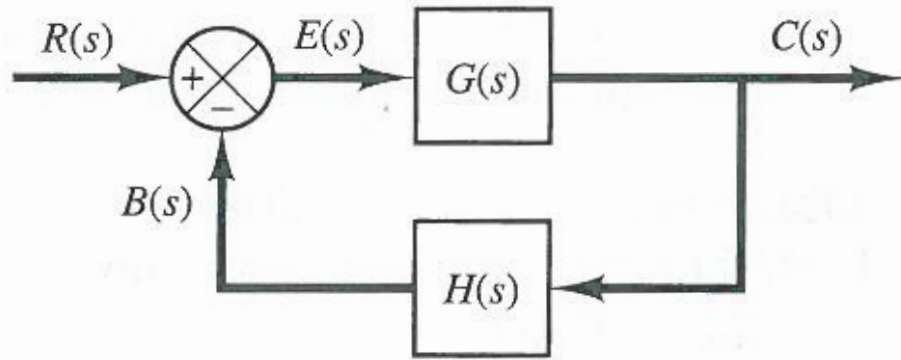


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sistema de lazo cerrado (con retroalimentación)



función de transferencia de lazo abierto:

$$\frac{B(s)}{E(s)} = G(s)H(s)$$

función de transferencia de lazo cerrado:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

$$C(s) = G(s)E(s)$$

$\Rightarrow$

$$E(s) = R(s) - B(s) = R(s) - H(s)C(s)$$