4 Convex sets. Separation

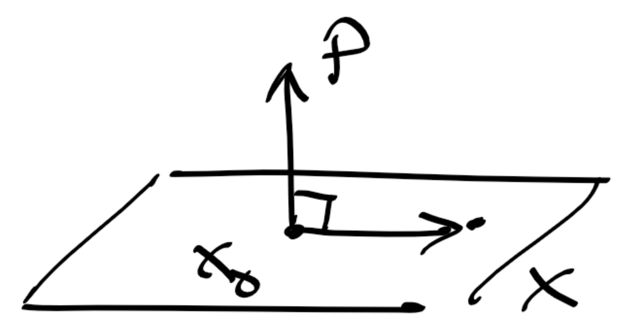
4.1 Review of topological concepts

DEFINITIONS. *Let $S \subseteq \mathbb{R}^n$.*

- ***Open ball*** *with centre* $\mathbf{a} \in \mathbb{R}^n$ *and radius* ε*:*

$$B_\varepsilon(\mathbf{a}) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{a}\| < \varepsilon\}$$

- *The complement of a set* S *is* $\complement S := \{\mathbf{a} \notin S\}$
- *A point* $\mathbf{a} \in \mathbb{R}^n$ *is an* ***interior point*** *of* S *iff* $\exists \varepsilon > 0 : B_\varepsilon(\mathbf{a}) \subseteq S$
- $\text{int}(S) := \{\mathbf{a} \text{ is interior point of } S\}$
- S *is* ***open*** *iff* $S = \text{int}(S)$
- *A point* $\mathbf{a} \in \mathbb{R}^n$ *is an* ***exterior point*** *of* S *iff* $\exists \varepsilon > 0 : B_\varepsilon(\mathbf{a}) \subseteq \complement S$
- *A point* $\mathbf{a} \in \mathbb{R}^n$ *is a* ***boundary point*** *of* S *iff any* $B_\varepsilon(\mathbf{a})$, *where* $\varepsilon > 0$, *contains points in both* S *and* $\complement S$
- $\partial S := \{\mathbf{a} \text{ is a boundary point of } S\}$
- *The* ***closure*** *of* S *is* $\text{cl}(S) := \text{int}(S) \cup \partial S$
- S *is* ***closed*** *iff* $S = \text{cl}(S)$, *i.e., iff* $\partial S \subseteq S$
- $S \subseteq \mathbb{R}^n$ *is* ***bounded*** *iff* $\exists R : S \subseteq B_R(\mathbf{0})$
- $S \subseteq \mathbb{R}^n$ *is* ***compact*** *iff it is closed and bounded*



Ex. A hyperplane $H = \{x \in \mathbb{R}^n : p^T(x - x_0) = 0\}$ has normal direction $p \neq 0$ and contains the point x_0. $\partial H = H$ closed.

A closed half-space: $p^T(x - x_0) \leq 0$

An open half-space: $p^T(x - x_0) < 0$

Lemma 1. *S is closed $\Leftrightarrow$ for any convergent sequence $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ in S, its limit point $\boldsymbol{x} \in S$*

Theorem 1 (Bolzano-Weierstrass). *Every sequence $\{\boldsymbol{x}_k\}_{k=1}^{\infty}$ in a compact set $S \subseteq \mathbb{R}^n$ has a subsequence $\{\boldsymbol{x}_k\}_{k \in \mathcal{K} \subseteq \mathbb{N}}^{\infty}$ which converges to a point in S.*

Theorem 2 (Weierstrass). *A continuous and real-valued function f defined on a compact set $S \subseteq \mathbb{R}^n$ attains its minimum and maximum, i.e., there is a point $\bar{\boldsymbol{x}} \in S$ such that $f(\bar{\boldsymbol{x}}) = \min_{x \in S} f(x)$ (and similarly for the max).*

Proof of Lemma 1: $\boxed{\Rightarrow}$ Assume S closed and $S \ni x_k \to x$ as $k \to \infty$. If $x \in \complement S$ open $\Rightarrow$ $\exists\, B_\varepsilon(x) \subseteq \complement S$ and there are infinitely many $x_k \in B_\varepsilon(x)$ and we have a contradiction, so $x \in S$.

$\boxed{\Leftarrow}$ Take any point $x \in \partial S$. For every $k \in \mathbb{N}$ take $x_k \in B_{1/k}(x) \cup S$. Then $x_k \to x$ as $k \to \infty$ and by assumption $x \in S$. Hence $\partial S \subseteq S$, and S is closed.

Ex. The half-space $p^T x \le p^T x_0$ is closed since any sequence stays in it: If $x_k \to \bar{x}$

$$\underbrace{p^T x_k}_{\text{continuous fcn}} \le p^T x_0 \quad \longrightarrow \quad p^T \bar{x} \le p^T x_0$$

Similarly for a system of inequalities:

$$\begin{cases} a_1^T x \le b_1 \\ a_2^T x \le b_2 \\ \vdots \\ a_m^T x \le b_m \end{cases} \Longleftrightarrow \begin{pmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{pmatrix} x \le \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \Longleftrightarrow Ax \le b$$

($\le$ in $Ax \le b$: vector ineq. (interpet it componentwise))

The polyhedral set

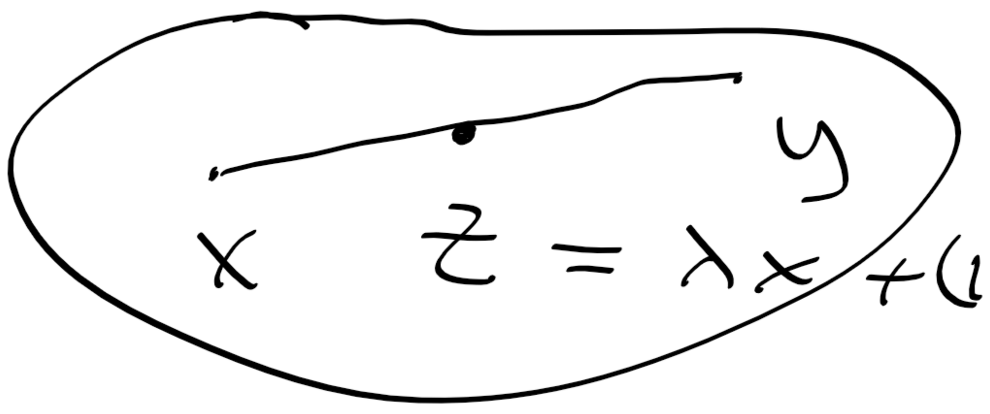
$$\{x \in \mathbb{R}^n : Ax \le b\} = \bigcap_{i=1}^{m} \{x \in \mathbb{R}^n ; a_i^T x \le b_i\}$$

intersection of closed half-spaces

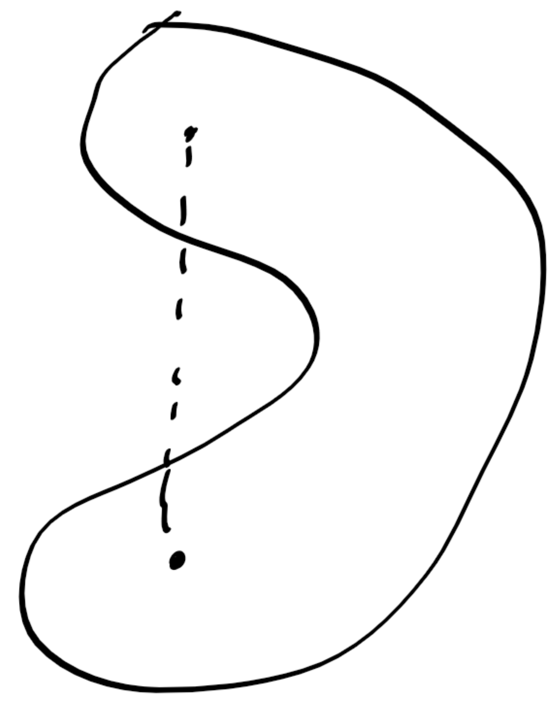
4.2 Convexity

Def. A set $S \subseteq \mathbb{R}^n$ is convex iff

$$\begin{cases} x, y \in S \\ 0 < \lambda < 1 \end{cases} \Longrightarrow \lambda x + (1-\lambda) y \in S$$



$z = \lambda x + (1-\lambda) y = y + \lambda (x - y)$



nonconvex

Ex. A polyhedral set

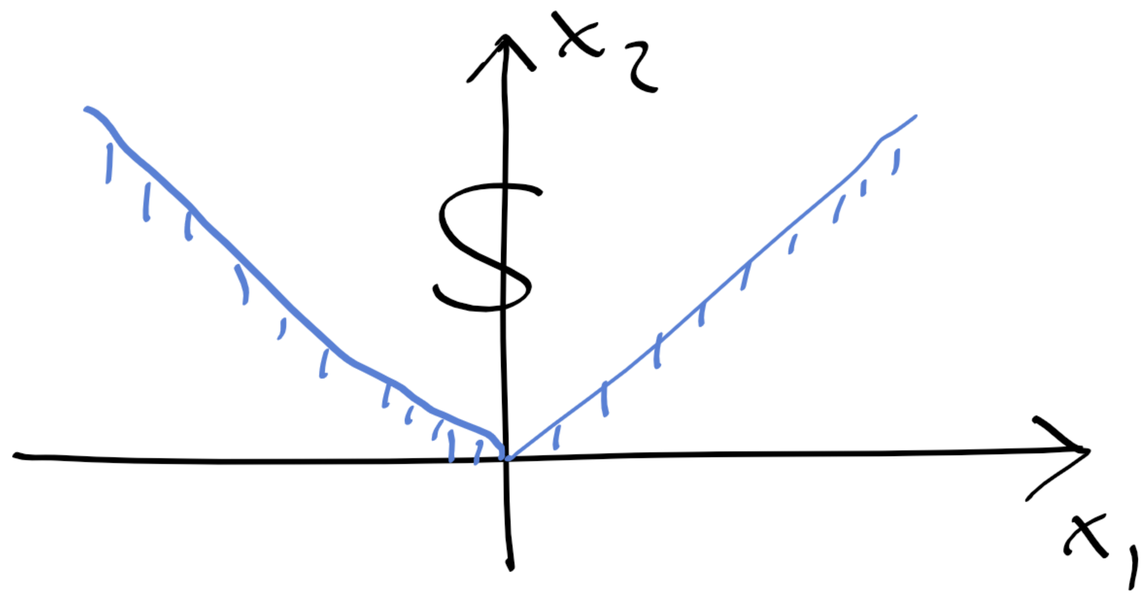
$P = \{ x \in \mathbb{R}^n : A x \leq b \}$ is convex:

$$\begin{cases} x, y \in P \\ 0 < \lambda < 1 \end{cases} \Rightarrow \begin{cases} A x \leq b \\ A y \leq b \\ 0 < \lambda < 1 \end{cases} \Rightarrow A(\lambda x + (1-\lambda) y) =$$

$$\lambda A x + (1-\lambda) A y \leq \lambda b + (1-\lambda) b = b \Rightarrow \lambda x + (1-\lambda) y \in P$$

Ex. $S = \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 \geq |x_1| \}$ is convex.

Proof 1: $\begin{cases} (x_1, x_2), (y_1, y_2) \in S \\ 0 < \lambda < 1 \end{cases} \Rightarrow$



$$\begin{cases} x_2 \geq |x_1| \\ y_2 \geq |y_1| \\ 0 < \lambda < 1 \end{cases} \Rightarrow \begin{cases} \lambda x_2 \geq \lambda |x_1| \\ (1-\lambda) y_2 \geq (1-\lambda) |y_1| \\ 0 < \lambda < 1 \end{cases}$$

$$\Rightarrow \lambda x_2 + (1-\lambda) y_2 \geq \lambda |x_1| + (1-\lambda) |y_1| = |\lambda x_1| + |(1-\lambda) y_1|$$

$$\geq |\lambda x_1 + (1-\lambda) y_1|$$

↑ $\triangle$-inequality

$$\Rightarrow \lambda (x_1, x_2) + (1-\lambda)(y_1, y_2) \in S.$$

Proof 2: $x_2 \geq |x_1| \iff \begin{cases} x_2 \geq x_1 \\ x_2 \geq -x_1 \end{cases} \iff \begin{cases} x_1 - x_2 \leq 0 \\ -x_1 - x_2 \leq 0 \end{cases}$

$\iff Ax \leq 0$ with $A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$. Hence

$S = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : Ax \leq 0 \right\}$ is a polyhedral set and convex (see above).

Lemma 2a. S_1 and S_2 convex $\Rightarrow$ $S_1 \cap S_2$ convex

Proof: Exerc. 4.8.

Def. of convexity can be rewritten:

- $\begin{cases} x_1, x_2 \in S \\ 0 < \lambda < 1 \end{cases} \Rightarrow \lambda x_1 + (1-\lambda) x_2 \in S$

is equivalent to $(\lambda_1 := \lambda,\ \lambda_2 := 1-\lambda)$

- $\begin{cases} x_1, x_2 \in S \\ 0 < \lambda_1, \lambda_2 < 1 \end{cases} \Rightarrow \lambda_1 x_1 + \lambda_2 x_2 \in S$

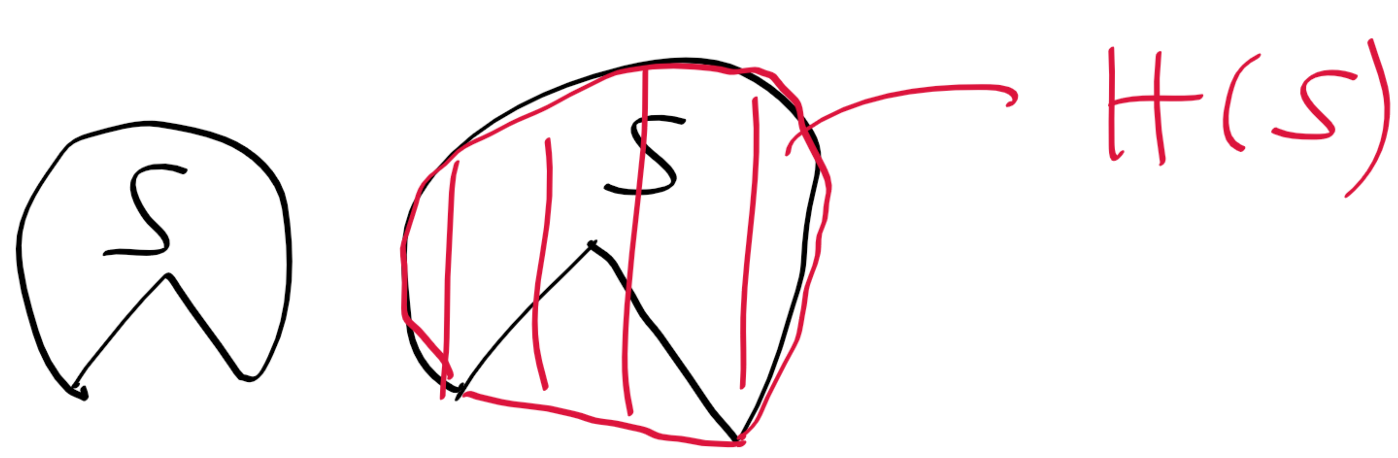
is equivalent to (Exerc. 4.9)

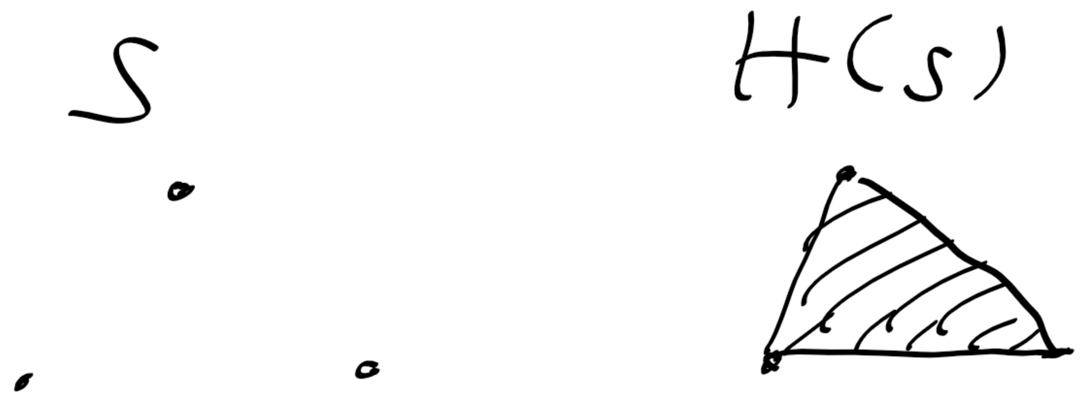
- $\begin{cases} x_1, \dots, x_k \in S \\ \lambda_i \geq 0,\ i = 1, \dots, k \\ \sum_1^k \lambda_i = 1 \end{cases} \Rightarrow \underbrace{\sum_{i=1}^{k} \lambda_i x_i}_{} \in S$

convex combination of vectors x_i

Def. The convex hull of a set $S \subseteq \mathbb{R}^n$ is $H(S) = \{$all convex combinations of elements in $S\}$

$= \left\{ \sum_{i=1}^{k} \lambda_i x_i \in \mathbb{R}^n : \sum \lambda_i = 1,\ \lambda_i \geq 0,\ \text{all } x_i \in S, k \in \mathbb{N} \right\}$

Ex. 

Ex. 

Ex. $S = \{ x \in \mathbb{R}^n : p^T x = \alpha \} \Rightarrow H(S) = S$

Lemma 3. $H(S)$ is convex.

Proof: Let $0<\lambda<1$ and $x, y \in H(S)$. Then

x is a convex combination of some $z_i \in S$

y ——— " ——— $z_j \in S$

Take all z_k. With some zero coefficients we can write

$$x = \sum_{k=1}^{m} \alpha_k z_k \quad \text{and} \quad y = \sum_{k=1}^{m} \beta_k z_k$$

with $\alpha_k, \beta_k \geq 0$ and $\sum \alpha_k = \sum \beta_k = 1$

Then

$$z_\lambda = \lambda x + (1-\lambda) y = \sum_{k=1}^{m} \underbrace{(\lambda \alpha_k + (1-\lambda)\beta_k)}_{\gamma_k \geq 0} \underbrace{z_k}_{\in S}$$

and $$\sum_{k=1}^{m} \gamma_k = \lambda \underbrace{\sum \alpha_k}_{1} + (1-\lambda) \underbrace{\sum \beta_k}_{1} = 1$$

Thus z_λ is a convex combination of $z_k \in S$ so that $z_\lambda \in H(S)$ #

Lemma 4. $$H(S) = \bigcap_{\substack{T \text{ convex} \\ T \supseteq S}} T$$

Proof: $\boxed{\supseteq}$ One of the $T = H(S)$.

$\boxed{\subseteq}$ $x \in H(S)$. Then $x \in \sum \lambda_i x_i$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$ and $x_i \in S \subseteq T$ for every T $\underset{T \text{ convex}}{\Rightarrow}$ $x \in T$ for every T

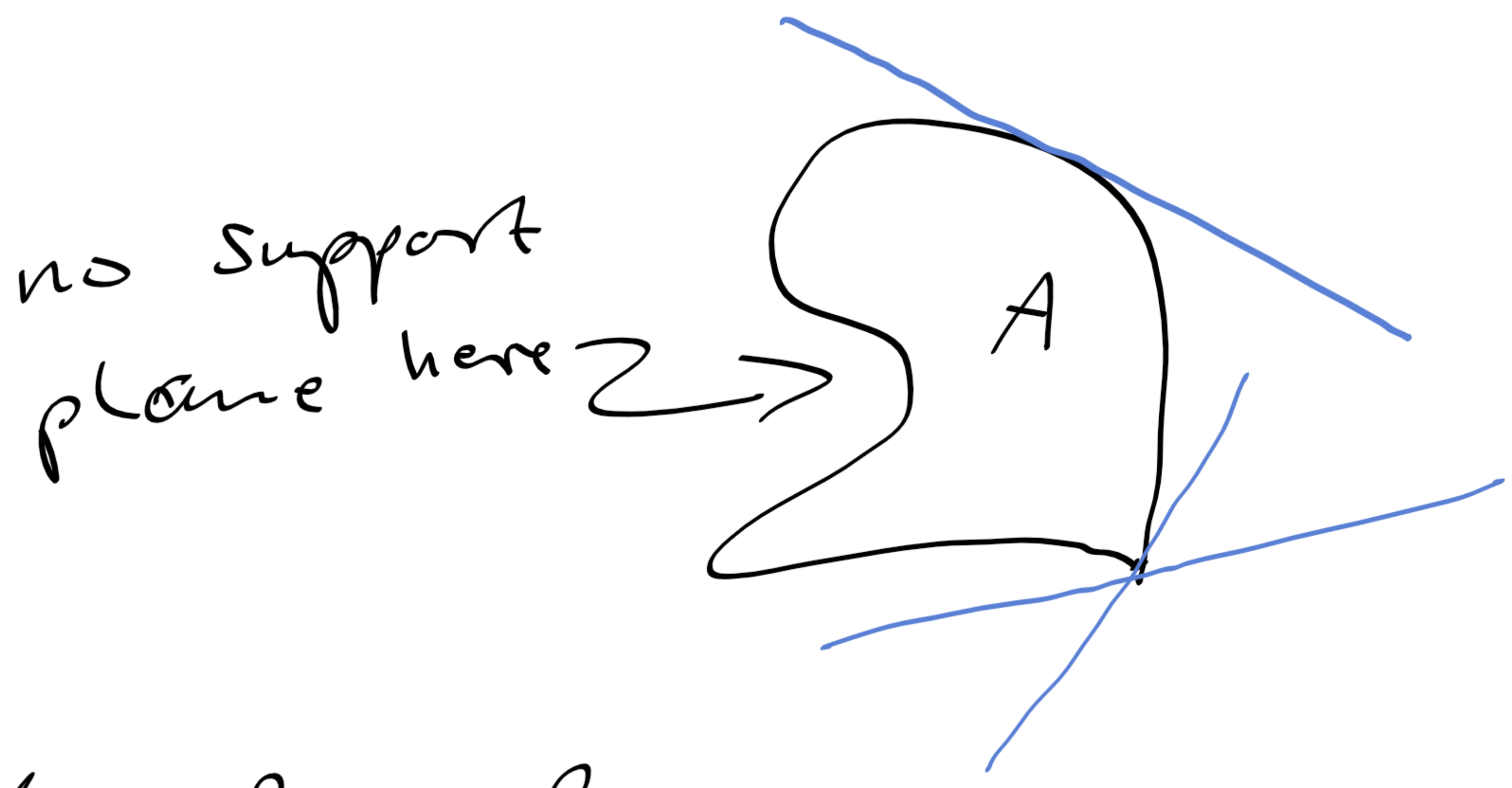
$$\Rightarrow x \in \bigcap_{\substack{T \text{ convex} \\ T \supseteq S}} T \quad \#$$

4.3 a Support planes

Def. 8. The hyperplane $p^T x = \alpha$ is a **support plane** to the set $A \subseteq \mathbb{R}^n$ iff

$$p^T x \leq \alpha \quad \forall x \in A$$

with equality for some $x \in \partial A$.



Thm 5: $\emptyset \neq S \subseteq \mathbb{R}^n$ closed and convex.

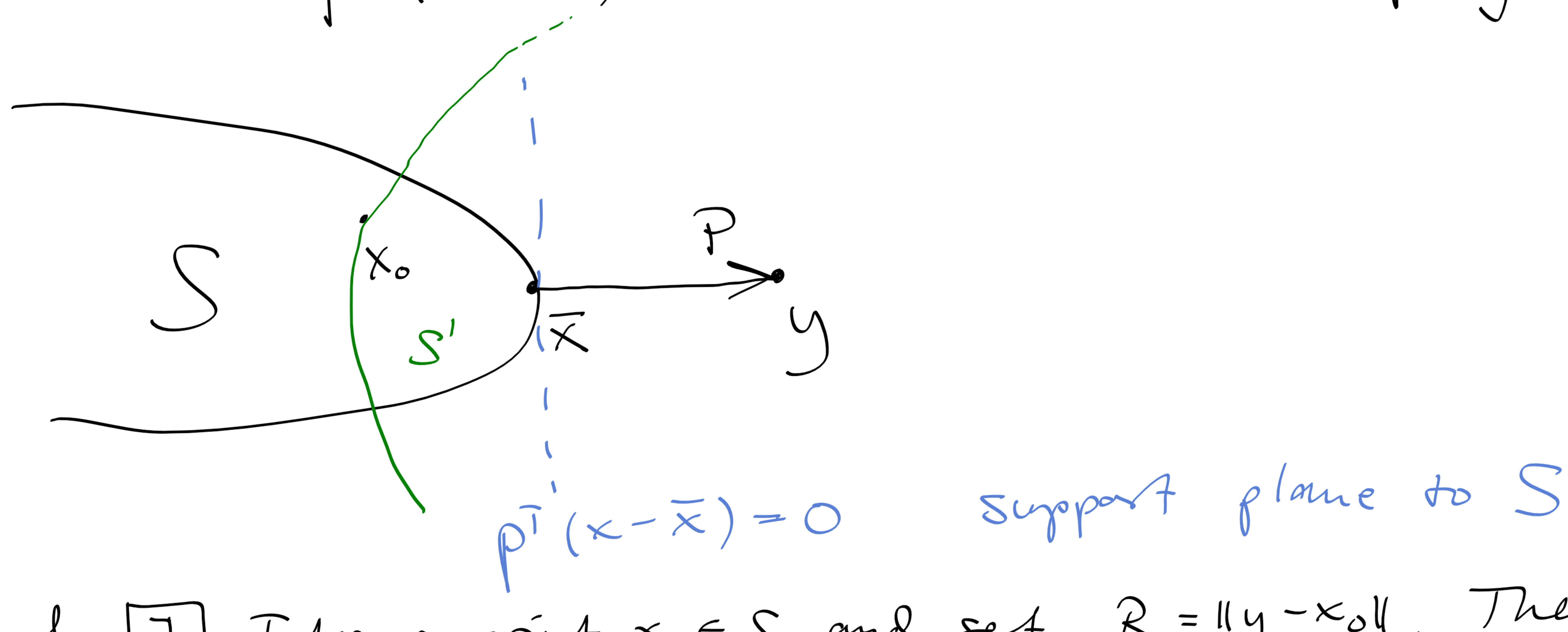
If $y \in \complement S$, then $\exists!\ \bar{x} \in S$ that solves

$$\underset{x \in S}{\text{minimize}} \|y - x\|$$

i.e. $\|y - \bar{x}\| = \min_{x \in S} \|y - x\| =: \text{dist}(y, S)$

Furthermore, $\bar{x}$ minimizer $\iff$

(*) $\quad p^T(x - \bar{x}) \leq 0 \quad \forall x \in S \quad$ where $p = y - \bar{x}$.



Proof: $\boxed{\exists}$ Take a point $x_0 \in S$ and set $R = \|y - x_0\|$. Then $S' = S \cap \{x \in \mathbb{R}^n : \|x - y\| \leq R\}$ is a compact set on which the continuous function $d(x) = \|y - x\|$ has a minimizer $\bar{x} \in S$ acc. to Weierstrass' thm.

Of course $\min_{x \in S} d(x) = \min_{x \in S'} d(x)$.

[(*)] $\bar{x}$ minimizer $\iff \|\underbrace{y - \bar{x}}_{p}\| \le \|y - x\| \quad \forall x \in S$

$$\iff \|p\|^2 \le \|y - x\|^2$$
$$= \|y - \bar{x} + \bar{x} - x\|^2$$
$$= \|p + (\bar{x} - x)\|^2 = (p + (\bar{x} - x))^T (p + (\bar{x} - x))$$
$$= \|p\|^2 + 2p^T(\bar{x} - x) + \|\bar{x} - x\|^2 \qquad \forall x \in S$$
$$\iff 2p^T(x - \bar{x}) \le \|x - \bar{x}\|^2 \quad \forall x \in S \qquad (**)$$

(*) $\to$ (**) is trivial. Conversely, replace x in (**) by $\lambda x + (1-\lambda)\bar{x} = \lambda(x - \bar{x}) + \bar{x} \in S \quad (0 < \lambda < 1)$ to get

$$2p^T(\lambda(x - \bar{x})) \le \|\lambda(x - \bar{x})\|^2 \iff$$
$$2p^T(x - \bar{x}) \le \lambda \|x - \bar{x}\|^2$$
$$\lambda \to 0 \implies 2p^T(x - \bar{x}) \le 0 \quad (*) \qquad (p = y - \bar{x})$$

[!] Assume $\hat{x}$ another minimizer. (*) gives

$$\begin{cases} (y - \bar{x})^T(x - \bar{x}) \le 0 \\ (y - \hat{x})^T(x - \hat{x}) \le 0 \end{cases} \qquad \forall x \in S$$

$$\implies \begin{cases} (y - \bar{x})^T(\hat{x} - \bar{x}) \le 0 \\ (y - \hat{x})^T(\bar{x} - \hat{x}) \le 0 \end{cases} \qquad \text{Add these:}$$

$$-\bar{x}^T(\hat{x} - \bar{x}) - \hat{x}^T(\bar{x} - \hat{x}) \le 0 \iff$$
$$\bar{x}^T(\bar{x} - \hat{x}) - \hat{x}^T(\bar{x} - \hat{x}) \le 0 \iff$$
$$(\bar{x} - \hat{x})^T(\bar{x} - \hat{x}) \le 0 \iff$$
$$\|\bar{x} - \hat{x}\|^2 \le 0 \iff \bar{x} = \hat{x}$$

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