4 Convex sets. Separation

4.1 Review of topological concepts

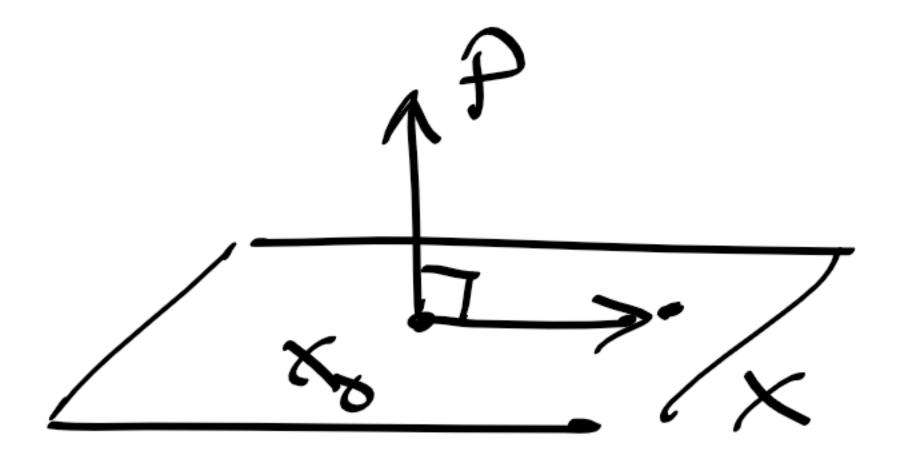
DEFINITIONS. Let $S \subseteq \mathbb{R}^n$.

• Open ball with centre $\mathbf{a} \in \mathbb{R}^n$ and radius $\boldsymbol{\varepsilon}$:

$$B_{\varepsilon}(\boldsymbol{a}) := \{ \boldsymbol{x} \in \mathbb{R}^n : ||\boldsymbol{x} - \boldsymbol{a}|| < \varepsilon \}$$

- The complement of a set S is $CS := \{a \notin S\}$
- A point $\mathbf{a} \in \mathbb{R}^n$ is an interior point of S iff $\exists \varepsilon > 0 : B_{\varepsilon}(\mathbf{a}) \subseteq S$
- $int(S) := \{a \text{ is interior point of } S\}$
- S is open iff S = int(S)
- A point $\mathbf{a} \in \mathbb{R}^n$ is an **exterior point** of S iff $\exists \varepsilon > 0 : B_{\varepsilon}(\mathbf{a}) \subseteq \mathbb{C}S$
- A point $\mathbf{a} \in \mathbb{R}^n$ is a **boundary point** of S iff any $B_{\varepsilon}(\mathbf{a})$, where $\varepsilon > 0$, contains points in both S and $\mathbb{C}S$
- $\partial S := \{a \text{ is a boundary point of } S\}$
- The **closure** of S is $cl(S) := int(S) \cup \partial S$
- S is **closed** iff S = cl(S), i.e., iff $\partial S \subseteq S$
- $S \subseteq \mathbb{R}^n$ is **bounded** iff $\exists R : S \subseteq B_R(\mathbf{0})$





Ex. A hyperplane
$$H = \{x \in \mathbb{R}^n : p^T(x-x_0) = 0\}$$

has normal direction $p \neq 0$ and contains
the point x_0 . $\exists H = H$ closed.
A closed half-space: $p^T(x-x_0) \leq 0$
An open half-space: $p^T(x-x_0) \leq 0$

LEMMA 1. S is closed \Leftrightarrow for any convergent sequence $\{x_k\}_{k=1}^{\infty}$ in S, its limit point $x \in S$

Theorem 1 (Bolzano-Weierstrass). Every sequence $\{x_k\}_{k=1}^{\infty}$ in a compact set $S \subseteq \mathbb{R}^n$ has a subsequence $\{x_k\}_{k \in \mathcal{K} \subset \mathbb{N}}^{\infty}$ which converges to a point in S.

Theorem 2 (Weierstrass). A continuous and real-valued function f defined on a compact set $S \subseteq \mathbb{R}^n$ attains its minimum and maximum, i.e., there is a point $\bar{\mathbf{x}} \in S$ such that $f(\bar{\mathbf{x}}) = \min_{x \in S} f(x)$ (and similarly for the max).

Proof of Lemma 1: [=] Assume S closed and $S \ni x_k \to x$ as $k \ni \infty$. If $x \in [S]$ open = $\exists B_{\varepsilon}(x) \subseteq [S]$ and there are infinitely many $x_k \in B_{\varepsilon}(x)$ and we have a contaction, so $x \in S$. [=] Take any point $x \in \partial S$. For every $k \in N$ take $x_k \in B_{1/k}(x) \cup S$. Then $x_k \to x$ as $k \to \infty$ and by assumption $x \in S$. Hence $\partial S \in S$, and $S \in S$ and $S \in S$.

Ex. The half-space pTx \le pTx \le pTx or is closed Time any sequence stays in it: If x => \f $P^{T}x_{u} \leq P^{T}x_{o} \rightarrow P^{T}x \leq P^{T}x_{o}$ continuous fch Similarly for a system of inequalities: $\begin{pmatrix}
a_1 T \times \leq b_1 \\
a_2 T \times \leq b_2
\end{pmatrix}$ $\begin{vmatrix}
a_1 T \\
a_m
\end{vmatrix}$ $\begin{vmatrix}
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\end{vmatrix}$ $\begin{vmatrix}
a_1 T \\
a_1 T \\$ The polyhedral set $\begin{cases} x \in \mathbb{R}^{n}: A \times \leq b \end{cases} = \bigcap_{i=1}^{n} \left\{ x \in \mathbb{R}^{n}: \alpha_{i}^{T} x \leq b_{i} \right\}$ intersection of dosed half-spaces

4.2 Converity Det. A set S=Rh is convex ift $\begin{cases} x, y \in S \\ \rightarrow \lambda \times + (1-\lambda)u \in S \end{cases}$ nonconvel Et. A polyhodral set $P = \{x \in \mathbb{R}^n : Ax \leq b\}$ is conver; $\lambda A \times + (1-\lambda) A y \leq \lambda b + (1-\lambda) b = b = 7 \lambda \times + (1-\lambda) y \in P$ \mathcal{G}_{X} , $S = \left\{ (x_1, x_2) \in \mathbb{R}^7 : x_2 \ge |X_1| \right\}$ is conver. Proof 1: $\begin{cases} (x_1, x_2), (y_1, y_2) \in S \\ 0 \le \lambda \le 1 \end{cases}$ $-\int \left\{ \begin{array}{l} \lambda \times_{2} \geq \lambda / x_{1} \\ (1-\lambda) y_{2} \geq (1-\lambda) |y_{1}| \\ oc \lambda < 1 \end{array} \right.$ $\begin{cases} x_2 \ge 1 \times_{i1} \\ y_2 \ge 1 y_{11} \end{cases}$ $\geq 1 \times 1 \times (1 \rightarrow)$ s-inequality

Proof 2:
$$x_1 \ge |x_1| \iff \begin{cases} x_1 \ge x_1 \\ x_1 \ge -x_1 \end{cases} \iff \begin{cases} x_1 - x_1 \le 0 \\ -x_1 - x_2 \le 0 \end{cases}$$

$$\Leftrightarrow A \times \le 0 \quad \text{with} \quad A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} \quad \text{Hence}$$

$$S = \begin{cases} X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : A \times = 0 \end{cases} \quad \text{is a polyhedral sex}$$
and connex (see above).

Lemma 2a. S, and S₂ connex $\Longrightarrow S_1 \cap S_2$ councel proof: $2x = x_1 - 4 - 8$

Dof of convexity can be rewritten.

$$\begin{cases} x_1, x_2 \le S \\ 0 \le A \le S \end{cases} \quad \lambda x_1 + (x - A)x_2 \in S$$

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$$\begin{cases} x_1, x_2 \in S \\ 0 \le A \le S \end{cases} \quad \lambda x_1 + A_2 x_2 \in S \end{cases}$$

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$$\begin{cases} x_1, x_2 \in S \\ 0 \le A \le S \end{cases} \quad \lambda x_2 + A_2 x_3 \in S \end{cases}$$

$$\begin{cases} x_1, x_2 \in S \\ 0 \le A \le S \end{cases} \quad \lambda x_1 + A_2 x_2 \in S \end{cases}$$

$$\begin{cases} x_1, x_2 \in S \\ 0 \le A \le S \end{cases} \quad \lambda x_2 \in S \end{cases} \quad \lambda x_3 \in S \end{cases}$$

$$\begin{cases} x_1, x_2 \in S \\ x_3 \in S \end{cases} \quad \lambda x_4 \in S \end{cases} \quad \lambda x_4 \in S \end{cases}$$

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$$\begin{cases} x_1, x_2 \in S \\ x_3 \in S \end{cases} \quad \lambda x_4 \in S \end{cases} \quad \lambda x_5 \in S \end{cases}$$

Proof: Let ochel and $x,y \in H(s)$. Then x is a convex combined on of some $z_i \in S$ y Take all z_k . With some zero coefficients we can write $x = \sum_{k=1}^{N} \alpha_{k} z_{k}$ and $y = \sum_{k=1}^{N} \beta_{k} z_{k}$ with α_{k} , $\beta_{k} \geq 0$ and $z_{k} = \sum_{k=1}^{N} \beta_{k} = 1$ Then $z_{j} = \lambda x + (1 - \lambda)y = \sum_{k=1}^{N} \frac{(\lambda \alpha_{k} + (-\lambda)\beta_{k})}{y_{k} \geq 0} = 1$ Proof: $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ Then $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ Then $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ Then $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ Proof: $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ Then $z_{j} = \lambda z_{j} = \lambda z_{j} = 1$ The	Lemma 3. H(s) is couver.
Take all Ξ_{k} . With some zero coefficients we can write $x = \sum_{k=1}^{\infty} \alpha_{k} \Xi_{k} \text{and} y = \sum_{k=1}^{\infty} \beta_{k} \Xi_{k}$ with α_{k} , $\beta_{k} \ge 0$ and $\Xi_{k} = \Xi_{k} \beta_{k} = 1$ Then $Z_{3} = \lambda x + k - \lambda y = \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ and $\sum_{k=1}^{\infty} Y_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ for the $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ so that $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ for the $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k}) \Xi_{k}$ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from $X_{k} = \lambda \sum_{k=1}^{\infty} (\lambda x_{k} + k - \lambda) \beta_{k} $ from	Proof: Let ochel and x,y e H(s). Then
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we can write $x = \sum_{k=1}^{m} \alpha_k E_k \text{and} y = \sum_{k=1}^{m} \beta_k E_k$ with $\alpha_k, \beta_k \ge 0$ and $\sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k) E_k$ $E_{\lambda} = \lambda x + k - \lambda) y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k) E_k$ and $\sum_{k=1}^{m} \gamma_k = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Then $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Then $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Then $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Then $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Then $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k - \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k + \lambda y = \sum_{k=1}^{m} (\lambda \alpha_k + k - \lambda) \beta_k = 1$ Thus $E_{\lambda} = \lambda x + k + \lambda y = 1$ Thus $E_{\lambda} = \lambda x + k + \lambda y = 1$ Thus	y 11 3; ∈S
$x = \sum_{k=1}^{\infty} \alpha_{k} Z_{k} \text{and} y = \sum_{k=1}^{\infty} \beta_{k} Z_{k}$ with α_{k} , $\beta_{k} \ge 0$ and $\sum_{k=1}^{\infty} \beta_{k} = 1$ Then $Z_{j} = \lambda x + k - \lambda y = \sum_{k=1}^{\infty} (\lambda \alpha_{k} + k - \lambda) \beta_{k} Z_{k}$ and $\sum_{k=1}^{\infty} \gamma_{k} = \lambda \sum_{k=1}^{\infty} (\lambda \alpha_{k} + k - \lambda) \beta_{k} Z_{k}$ and $\sum_{k=1}^{\infty} \gamma_{k} = \lambda \sum_{k=1}^{\infty} (\lambda \alpha_{k} + k - \lambda) \beta_{k} Z_{k}$ and $\sum_{k=1}^{\infty} \gamma_{k} = \lambda \sum_{k=1}^{\infty} (\lambda \alpha_{k} + k - \lambda) \beta_{k} Z_{k}$ From Z_{j} is a convex combination of $Z_{k} \in S_{k}$ so that $Z_{j} \in H(S)$ # Lemma 4. $H(S) = \bigcap_{k=1}^{\infty} \sum_{k=1}^{\infty} (\lambda \alpha_{k} + k - \lambda) \beta_{k} Z_{k}$ From Z_{j} is a convex combination of $Z_{k} \in S_{k}$ Formal Z_{j} is a convex combination of Z_{j} is a convex combination of Z_{j} is a convex combination of Z_{j} is a	Take all Zk. With some zero coefficient
Then $z_1 = \lambda x + k - \lambda y = \frac{m}{2} \left(\lambda x_k + k - \lambda y_k \right) z_k$ $z_1 = \lambda x + k - \lambda y = \frac{m}{2} \left(\lambda x_k + k - \lambda y_k \right) z_k$ and $\sum_{k=1}^{m} Y_k = \lambda \sum_{k=1}^{m} (\lambda x_k + k - \lambda y_k) z_k$ and $\sum_{k=1}^{m} Y_k = \lambda \sum_{k=1}^{m} (\lambda x_k + k - \lambda y_k) z_k$ Thus $z_1 = \lambda \sum_{k=1}^{m} (\lambda x_k + k - \lambda y_k) z_k$ So that $z_1 = \lambda \sum_{k=1}^{m} (\lambda x_k + k - \lambda y_k) z_k$ So that $z_1 = \lambda \sum_{k=1}^{m} (\lambda x_k + k - \lambda y_k) z_k$ Then $z_1 = \lambda x + k - \lambda y_k = z_k$ Then $z_1 = \lambda x_k + k - \lambda y_k +$	we can usside $\frac{m}{2}$ and $\frac{m}{2}$ $\frac{m}{2}$
Then $Z_{\lambda} = \lambda x + k - \lambda \lambda y = \frac{M}{2} \left(\lambda x_{k} + k - \lambda \right) \beta_{k} z_{k}$ and $\sum_{k=1}^{M} Y_{k} = \lambda \sum_{k=1}^{M} (\lambda x_{k} + k - \lambda) \beta_{k} z_{k} z_{k}$ and $\sum_{k=1}^{M} Y_{k} = \lambda \sum_{k=1}^{M} (\lambda x_{k} + k - \lambda) \beta_{k} z_{k} z_{k}$ Thus Z_{λ} is a convex combination of $Z_{k} \in S_{k}$ so that $Z_{\lambda} \in H(S)$ # Lemma 4. $H(S) = \prod_{k=1}^{M} \sum_{k=1}^{M} (\lambda x_{k} + k - \lambda) \beta_{k} z_{k} z_{k}$ Proof: $Z_{\lambda} = \lambda x_{k} + k - \lambda y_{k} z_{k} z_{k} z_{k}$ Tonver $Z_{\lambda} = \lambda x_{k} + k - \lambda y_{k} z_{k} z$	
$Z_{\lambda} = \lambda x + (1 - \lambda)y = \frac{1}{2} \left(\lambda x_{k} + (1 - \lambda)\beta_{k} \right)^{2} \lambda $ and $\sum_{k=1}^{M} Y_{k} = \lambda \sum_{k=1}^{M} (1 - \lambda)\sum_{k=1}^{M} \beta_{k} = 1$ Thus Z_{λ} is a convex combination of Z_{k} essentially so that $Z_{\lambda} \in H(S) \qquad \text{the } T = H(S)$ Fund: $\sum_{k=1}^{M} One \text{of } \text{the } T = H(S).$ $\sum_{k=1}^{M} A_{k} \in H(S). \text{ Then } x \in \mathbb{Z}^{1} \setminus X_{k}, \text{if } x \in \mathbb{Z}^{$	
and $\sum_{k=1}^{m} Y_k = \lambda \sum_{k=1}^{m} x_k + (1-\lambda) \sum_{j=1}^{m} y_k = 1$ Thus Z_{λ} is a convex combaination of $Z_{k} \in S_{k}$ so that $Z_{\lambda} \in H(S)$ # Lemma 4. $H(S) = \bigcap_{j=1}^{m} T_{j}$ Final $T_{j} \in H(S_{j})$ Proof: $T_{j} \in H(S_{j})$ $T_{j} \in H(S_{j})$ Then $T_{j} \in T_{j}$ and $T_{j} \in S_{j}$ Then $T_{j} \in T_{j}$	$\frac{1}{2} - \lambda x + (1 - \lambda) u = \frac{1}{2} \left(\lambda x_{\mu} + (1 - \lambda) \beta_{\mu} \right)^{2} \lambda$
So that $Z_{\lambda} \in H(S)$ # Lemma 4. $H(S) = \bigcap T$ Toonver $T \ge S$ Proof: $\supseteq One \ of \ the \ T = H(S)$. $\boxtimes \times \in H(S)$. Then $\times \in \mathbb{Z} \lambda_{i} \times_{i}$, $\mathbb{Z} \lambda_{i} = 1, \lambda_{i} > 0$ and $\times_{i} \in S \subseteq T$ for every $T \longrightarrow \times \in T$ for T Then $T \in \mathbb{Z} \setminus X$	and $\sum_{k=1}^{m} Y_k = \lambda \sum_{k=1}^{\infty} X_k + (1-\lambda) \sum_{k=1}^{\infty} \beta_k = 1$
Lemma 4. $H(S) = \bigcap T$ The convert $T \ge S$ Proof: \square One of the $T = H(S)$. \square	Thus Zy is a convex combination of Zhes
Proof: \supseteq One of the $T = H(s)$. $\supseteq X \in H(s)$. Then $X \in \supseteq \lambda_1 \times i_1$, $\supseteq \lambda_1 = 1, \lambda_2 = 1$ and $X \in S \subseteq T$ for every $I \longrightarrow X \in I$ for $I \subseteq I$ $I \subseteq$	so that $Z_{\lambda} \in H(S)$ #
Proof: \supseteq One of the $T = H(s)$. $\boxtimes X \in H(s)$. Then $X \in \mathbb{Z} : X_i :$	
	$r \geq S$
and $K: ESET$ for every $T \longrightarrow XET$ for T	Proof: Done of the T=H(s).
\rightarrow $\times \in \Omega T +$	
\rightarrow $\times \in \Omega T +$	and K; ESEI for every 1 => XEI for Tornvex every T

7>5

4.3 a Support planes Def. 8. The hyperplane $p^Tx = x$ is a support plane to the set $A \subseteq \mathbb{R}^n$ iff $p^{T}x \leq \infty$ $\forall x \in A$ with equality for some x ∈ DA. no support plane here Z Thm 5: \$ + S = R" closed and conver. If yels, then I! xeS that solves minimize || y - x|| x = S i.e. $\|y-\bar{x}\| = \min_{x \in S} \|y-\bar{x}\| = : \text{dist}(y, S)$ Fur the more, x minimizer (=) $(x) \qquad p^{T}(x-\overline{x}) \leq 0 \qquad \forall x \in S \quad \text{where } p = y - \overline{x}$ $\int_{\overline{p}} (x - \overline{x}) = 0$ Support plane to S Proof: [] Take a point $x_0 \in S$ and set $R = ||y - x_0||$. Then $5' = 5 \cap \{x \in \mathbb{R}^n : \|x - y\| \le \mathbb{R} \}$ is a compact set on which the continuous function $d(x) = \|y - x\|$ has a uniminizer $x \in S$ acc. to Weierstass'thm.

```
Of course mind(x) = min d(x).

x \in S x \in S'
   \overline{x} \quad 
                  = 1 \quad || p ||^2 \leq || y - x ||^2
                                                                                                                                            = 119 - \overline{x} + \overline{z} - \times 11
                                                                                                                                         = \| \rho + (\overline{x} - x) \|^2 = (\rho + (\overline{x} - x))^{\top} (\rho + (\overline{x} - x))
                                                                                                                                               = \|\rho\|^2 + 2 \rho^7 (\bar{x} - x) + \|(\bar{x} - x)\|^2
                                                                                                             2\rho^{T}(x-\overline{x}) \leq \|x-\overline{x}\|^{2} \quad \forall x \in S \quad (xx)
                       (x) -) (xx) is trivial. Conversely, replace x in (xx)
                         by \lambda \times + (1-\lambda) \overline{\times} = \lambda(x-\overline{x}) + \overline{x} \in S (02\lambda21) to get
                                                                                                         2pT(\lambda(x-\bar{x})) \leq ||\lambda(x-\bar{x})||^{\zeta} \qquad (=)
                                                                                                           2\rho^{T}(x-\overline{x}) \leq \lambda \|x-\overline{x}\|^{2}
                                                                                                                                                                                                                                                                                                                                                                                                      \left(\rho = y - \bar{z}\right)
                                 \lambda \rightarrow 0 \Rightarrow 2pT(x-\overline{x}) \leq 0 \tag{*}
                  II Assume à another minimizer.
                                                                                                                                                                                                                                                                                                                                                                                                                                           (X) Sive
                                                                                                  \begin{cases} (y - \overline{x})^T (x - \overline{x}) \leq 0 \\ (y - \overline{x})^T (x - \overline{x}) \leq 0 \end{cases}
                                                                                                                                                                                                                                                                                                                            \forall \times \in \mathcal{S}
                                                                                                           \int (y - \overline{x})^T (\hat{x} - \overline{x}) \leq 0
(y - \hat{x})^T (\overline{x} - \hat{x}) \leq 0
                                                                                                                                                                                                                                                                                                                                                                                        Add these:
                                                                 -\overline{x}T(x^{2}-\overline{x})-x^{2}(\overline{x}-x^{2})\leq 0
                                                                                   \frac{1}{x} \left( \frac{1}{x} - \frac{1}{x} \right) - \frac{1}{x} \left( \frac{1}{x} - \frac{1}{x} \right) \leq 0 
                                                                                    \left(\frac{1}{x} - \hat{x}\right)^{T} \left(\frac{1}{x} - \hat{x}\right) \leq 0 \tag{(=)}
                                                                                               ||x-x||^2 \leq 0 \quad (=) \quad \overline{x} = \hat{x}
                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                             +
```