"When a flow is both frictionless and irrotational, pleasant things happen."

We can treat external flows around bodies as *invicid* (i.e. frictionless) and *irrotational* (i.e. the fluid particles are not rotating). This is because the viscous effects are limited to a thin layer next to the body called the boundary layer..

We can define a *potential function*(x, z, t) , as a continuous function that satisfies the basic laws of fluid mechanics: *conservation of mass and momentum*, assuming incompressible, inviscid and irrotational flow.

The Stream Function

Is a clever device which allows us to wipe out the continuity equation and solve the momentum equation directly for the single variable.

Continuity equation

Cartesian: $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

Cylindrical: $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0$$

The most common application is incompressible flow in the xy plane

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

This equation is satisfied identically if a function $\psi(x, y)$ is defined such that and the above equation becomes

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) \equiv 0$$

The Eq. shows that this new function must be defined such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\mathbf{V} = \mathbf{i}\,\frac{\partial \psi}{\partial y} - \mathbf{j}\,\frac{\partial \psi}{\partial x}$$

The vorticity, or curl V, is an interesting function

$$\text{curl } \mathbf{V} = 2\mathbf{k}\omega_z = -\mathbf{k}\nabla^2\psi \qquad \text{Where} \qquad \nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}$$

One important application is inviscid irrotational flow in the xy plane, where $\omega_z = 0$ and the above Eq. is reduced to

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

This is the second-order Laplace equation for which many solutions and analytical techniques are known. Also, boundary conditions like Eq. reduce to

At infinity:	$\psi = U_\infty y + \text{const}$
At the body:	$\psi = \text{const}$

Geometric Interpretation of Stream Function

The fancy mathematics above would serve by itself to make the stream function immortal and always useful to engineers. Even better, though, stream function has a beautiful geometric interpretation: Lines of constant stream function are streamlines of the flow. This can be shown as follows. The definition of a streamline in two-dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v}$$

$$u\,dy - v\,dx = 0 \qquad \text{streamline}$$

By the definition of stream function we obtain

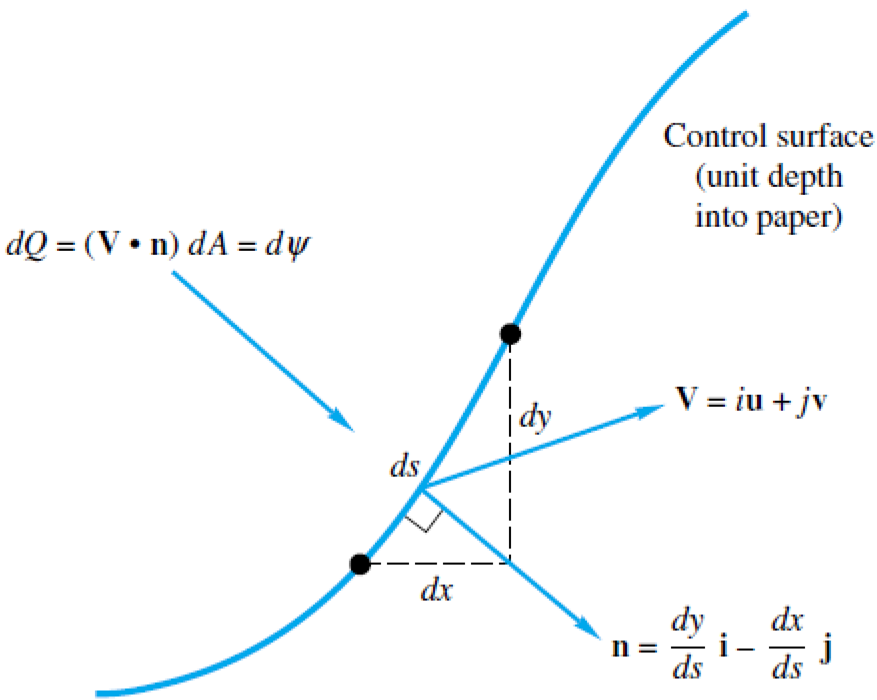
$$\frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy = 0 = d\psi$$

Thus the change in ψ is zero along a streamline, or

$$\psi = \text{const along a streamline}$$

Geometric Interpretation of Stream Function

Having found a given solution ψ(x, y), we can plot lines of constant to give the streamlines of the flow. There is also a physical interpretation which relates to volume flow. We can compute the volume flow dQ through an element ds of control surface of unit depth

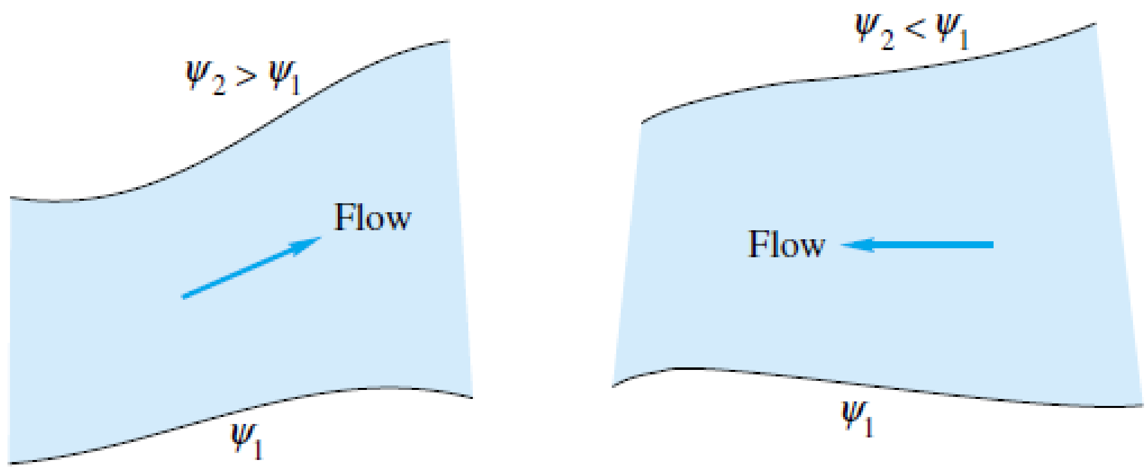


$$dQ = (\mathbf{V} \cdot \mathbf{n})\, dA = \left(\mathbf{i}\,\frac{\partial \psi}{\partial y} - \mathbf{j}\,\frac{\partial \psi}{\partial x}\right) \cdot \left(\mathbf{i}\,\frac{dy}{ds} - \mathbf{j}\,\frac{dx}{ds}\right) ds(1)$$

$$= \frac{\partial \psi}{\partial x}\, dx + \frac{\partial \psi}{\partial y}\, dy = d\psi$$

Geometric Interpretation of Stream Function

Sign convention for flow in terms of change in stream function: (a) flow to the right if ψ_U is greater; (b) flow to the left if ψ_L is greater.



Thus the change in ψ across the element is numerically equal to the volume flow through the element. The volume flow between any two points in the flow field is equal to the change in stream function between those points:

$$Q_{1\to 2} = \int_1^2 (\mathbf{V} \cdot \mathbf{n})\, dA = \int_1^2 d\psi = \psi_2 - \psi_1$$

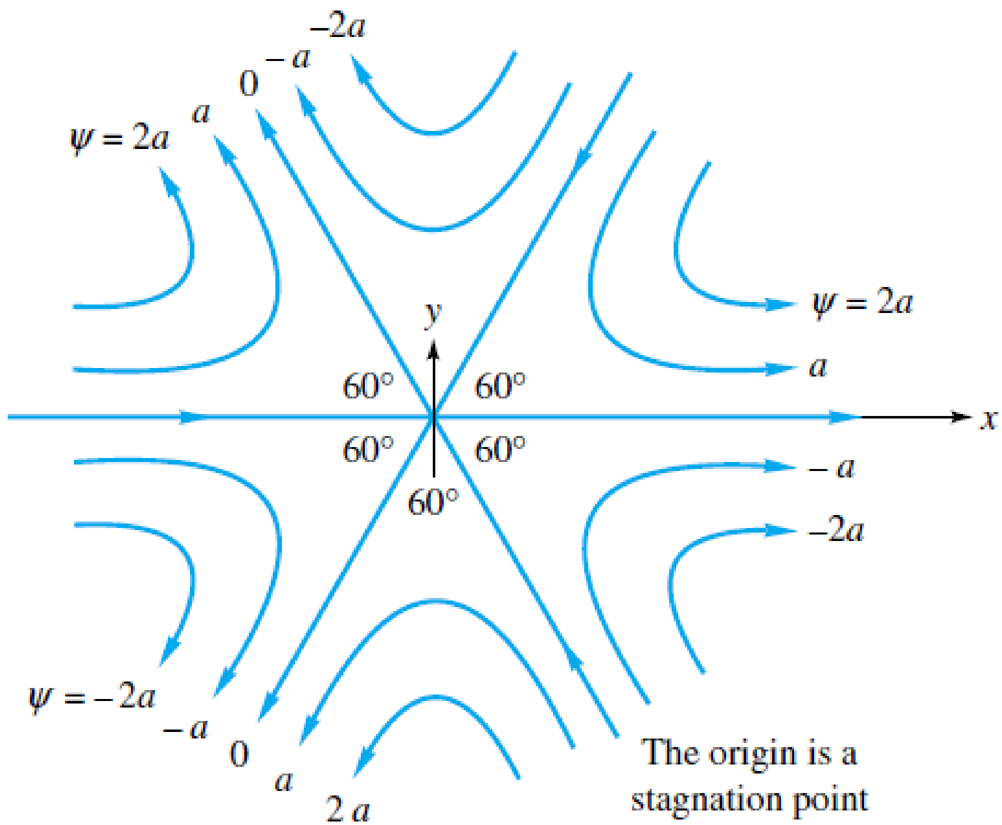
Geometric Interpretation of Stream Function

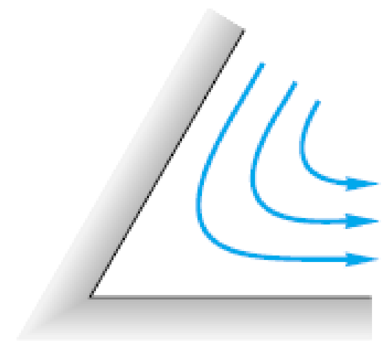
Example: If a stream function exists for the velocity field of

$$u = a(x^2 - y^2) \quad v = -2axy \quad w = 0$$

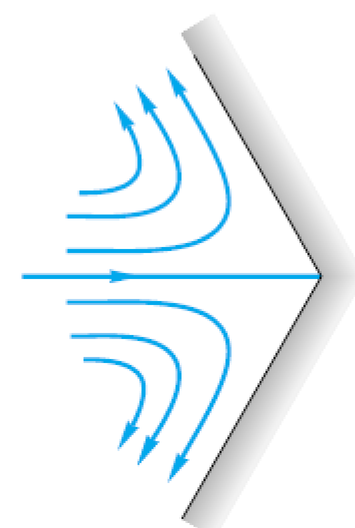
find it, plot it, and interpret it.

Geometric Interpretation of Stream Function





Flow around a 60° corner



Frictionless Irrotational Flow

When a flow is both frictionless and irrotational, pleasant things happen. First, the momentum equation reduces to Euler's equation

$$\rho \frac{d\mathbf{V}}{dt} = \rho \mathbf{g} - \boldsymbol{\nabla} p$$

Second, there is a great simplification in the acceleration term.

$$\frac{d\mathbf{V}}{dt} = \frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V}$$

A beautiful vector identity exists for the second term

$$(\mathbf{V} \cdot \boldsymbol{\nabla})\mathbf{V} \equiv \boldsymbol{\nabla}(\tfrac{1}{2}V^2) + \boldsymbol{\zeta} \times \mathbf{V} \qquad \text{where } \boldsymbol{\zeta} = \text{curl } \mathbf{V}$$

Frictionless Irrotational Flow

Divide by ρ, and rearrange on the left-hand side. The entire equation into an arbitrary vector displacement dr:

$$\left[\frac{\partial \mathbf{V}}{\partial t} + \boldsymbol{\nabla}\left(\frac{1}{2} V^2\right) + \boldsymbol{\zeta} \times \mathbf{V} + \frac{1}{\rho} \boldsymbol{\nabla} p - \mathbf{g}\right] \cdot d\mathbf{r} = 0$$

Nothing works right unless we can get rid of the third term. We want

$$(\boldsymbol{\zeta} \times \mathbf{V}) \cdot (d\mathbf{r}) \equiv 0$$

This will be true under various conditions:

1. V is zero; trivial, no flow (hydrostatics).
2. ξ is zero; irrotational flow.
3. dr is perpendicular to (ξ x V) ; this is rather specialized and rare.
4. dr is parallel to V; we integrate along a streamline

Frictionless Irrotational Flow

The last condition is the common assumption. If we integrate along a streamline in frictionless compressible flow and take, for convenience, **g=´-gk,**

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} + d\left(\frac{1}{2} V^2\right) + \frac{dp}{\rho} + g\, dz = 0$$

Except for the first term, these are exact differentials. Integrate between any two points 1 and 2 along the streamline:

$$\int_1^2 \frac{\partial V}{\partial t}\, ds + \int_1^2 \frac{dp}{\rho} + \frac{1}{2}\left(V_2^2 - V_1^2\right) + g(z_2 - z_1) = 0$$

The above equation is Bernoulli’s equation for frictionless unsteady flow along a streamline. For incompressible steady flow, it reduces to

$$\frac{p}{\rho} + \frac{1}{2} V^2 + gz = \text{constant along streamline}$$

Velocity Potential

Irrotationality gives rise to a scalar function similar and complementary to the stream Function. A vector with zero curl must be the gradient of a scalar function

$$\text{If} \qquad \nabla \times \mathbf{V} \equiv 0 \qquad \text{then} \qquad \mathbf{V} = \nabla\phi$$

where ϕ =ϕ (x, y, z, t) is called the velocity potential function. Knowledge of ϕ thus immediately gives the velocity components

$$u = \frac{\partial\phi}{\partial x} \qquad v = \frac{\partial\phi}{\partial y} \qquad w = \frac{\partial\phi}{\partial z}$$

Note that ϕ, unlike the stream function, is fully three-dimensional and not limited to two coordinates. It reduces a velocity problem with three unknowns u, v, and w to a single unknown potential ϕ. The velocity potential also simplifies the unsteady Bernoulli equation because if ϕ exists, we obtain

$$\frac{\partial \mathbf{V}}{\partial t} \cdot d\mathbf{r} = \frac{\partial}{\partial t}(\nabla\phi) \cdot d\mathbf{r} = d\left(\frac{\partial\phi}{\partial t}\right)$$

Velocity Potential

Unsteady Bernoulli Equation then becomes a relation between ϕ and ρ

$$\frac{\partial \phi}{\partial t} + \int \frac{dp}{\rho} + \frac{1}{2} \left| \nabla \phi \right|^2 + gz = \text{const}$$

This is the unsteady irrotational Bernoulli equation. It is very important in the analysis of accelerating flow fields

Orthogonality of Streamlines and Potential Lines

If a flow is both irrotational and described by only two coordinates, ψ and ϕ both exist and the streamlines and potential lines are everywhere mutually perpendicular except at a stagnation point. For example, for incompressible flow in the xy plane, we would have

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \phi}{\partial x}$$

Cauchy-Riemann equations

$$\upsilon = -\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$$

Velocity Potential

Generation of Rotationality

We have discussed Bernoulli's equation under different circumstances. Such reinforcement is useful, This equation is probably the most widely used equation in fluid mechanics.

- It requires frictionless flow with no shaft work or heat transfer between sections 1 and 2.
- The flow may or may not be irrotational,

The only remaining question is: *When* is a flow irrotational?
In other words, when does a flow have negligible angular velocity?

The exact analysis of fluid rotationality under arbitrary conditions is a topic for advanced study,

Velocity Potential

Generation of Rotationality

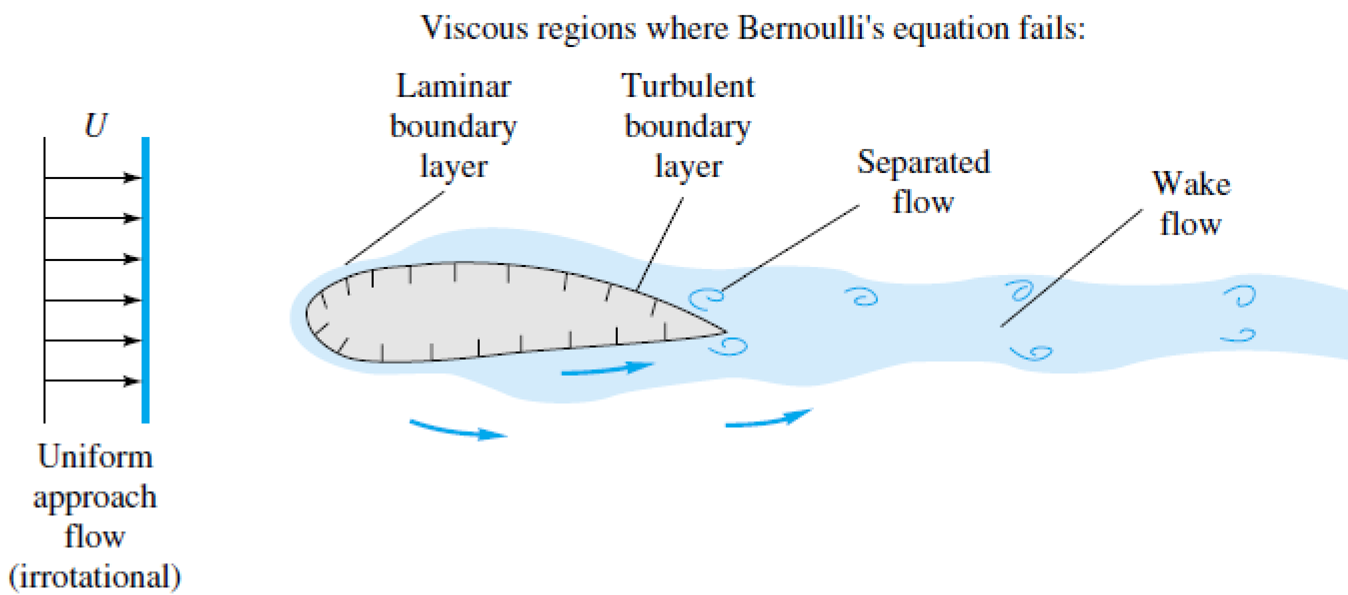
A fluid flow which is initially irrotational may become rotational if

- High viscous forces induced by jets, wakes, or solid boundaries. In this situation Bernoulli's equation will not be valid in such viscous regions.
- There are entropy gradients caused by curved shock waves.
- There are density gradients caused by stratification (uneven heating) rather than by pressure gradients.
- There are significant noninertial effects such as the earth's rotation (the Coriolis acceleration).

Velocity Potential

Generation of Rotationality

A fluid flow which is initially irrotational may become rotational if

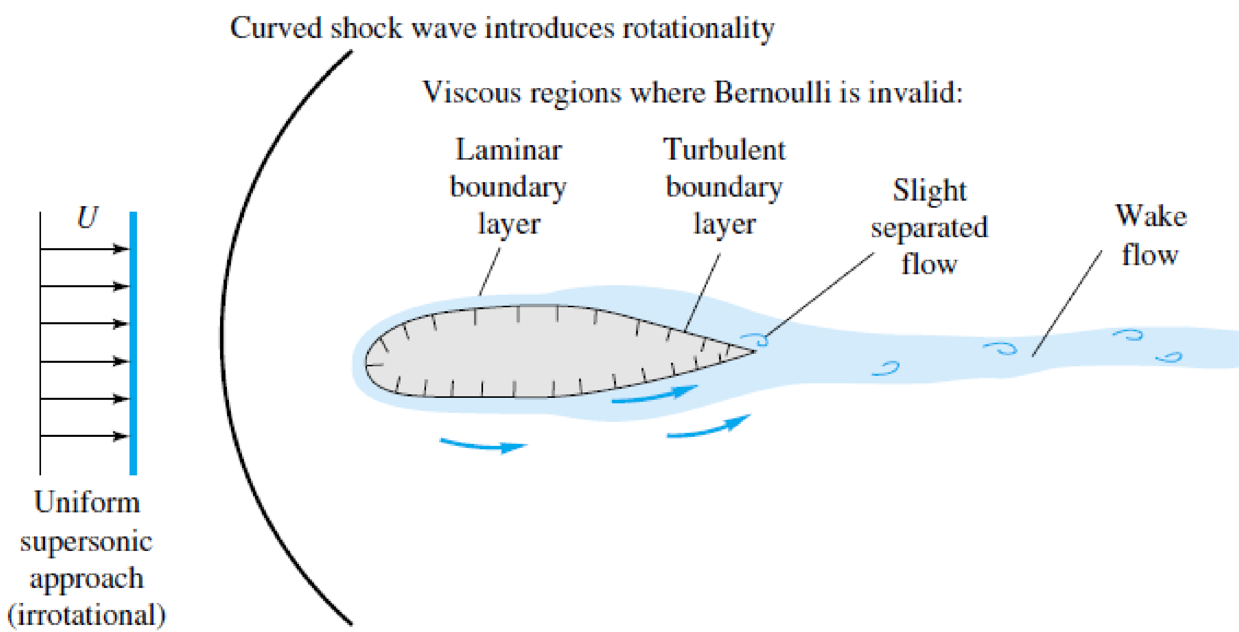


Typical flow patterns illustrating viscous regions patched into nearly frictionless regions:
(a) low subsonic flow past a body($U << a$); frictionless, irrotational potential flow outside the boundary layer (Bernoulli and Laplace equations valid)

Velocity Potential

Generation of Rotationality

A fluid flow which is initially irrotational may become rotational if



Supersonic flow past a body (U > a); frictionless, rotational flow outside the boundary layer (Bernoulli equation valid, potential flow invalid). the flow downstream is *rotational* due to entropy gradients. Euler’s equation can be used in this frictionless region but not potential theory.

Velocity Potential

Generation of Rotationality

A fluid flow which is initially irrotational may become rotational if

- Internal flows, such as pipes and ducts, are mostly viscous, and the wall layers grow to meet in the core of the duct. Bernoulli's equation does not hold in such flows unless it is modified for viscous losses.

- External flows, such as a body immersed in a stream, are partly viscous and partly inviscid, the two regions being patched together at the edge of the shear layer or boundary layer.

Velocity Potential

Generation of Rotationality

A fluid flow which is initially irrotational may become rotational if

The approach stream is irrotational; i.e., the curl of a constant is zero, but viscous stresses create a rotational shear layer beside and downstream of the body.

The shear layer is laminar, or smooth, near the front of the body and turbulent, or disorderly, toward the rear. A separated, region usually occurs near the trailing edge, followed by an unsteady turbulent wake extending far downstream.

Some sort of laminar or turbulent viscous theory must be applied to these viscous regions; they are then patched onto the outer flow, which is frictionless and irrotational.

If the stream Mach number is less than about 0.3, the fluid flow is consider incompressible.

$$\nabla \cdot \mathbf{V} = \nabla \cdot (\nabla \phi) = 0$$

$$\nabla^2 \phi = 0 = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

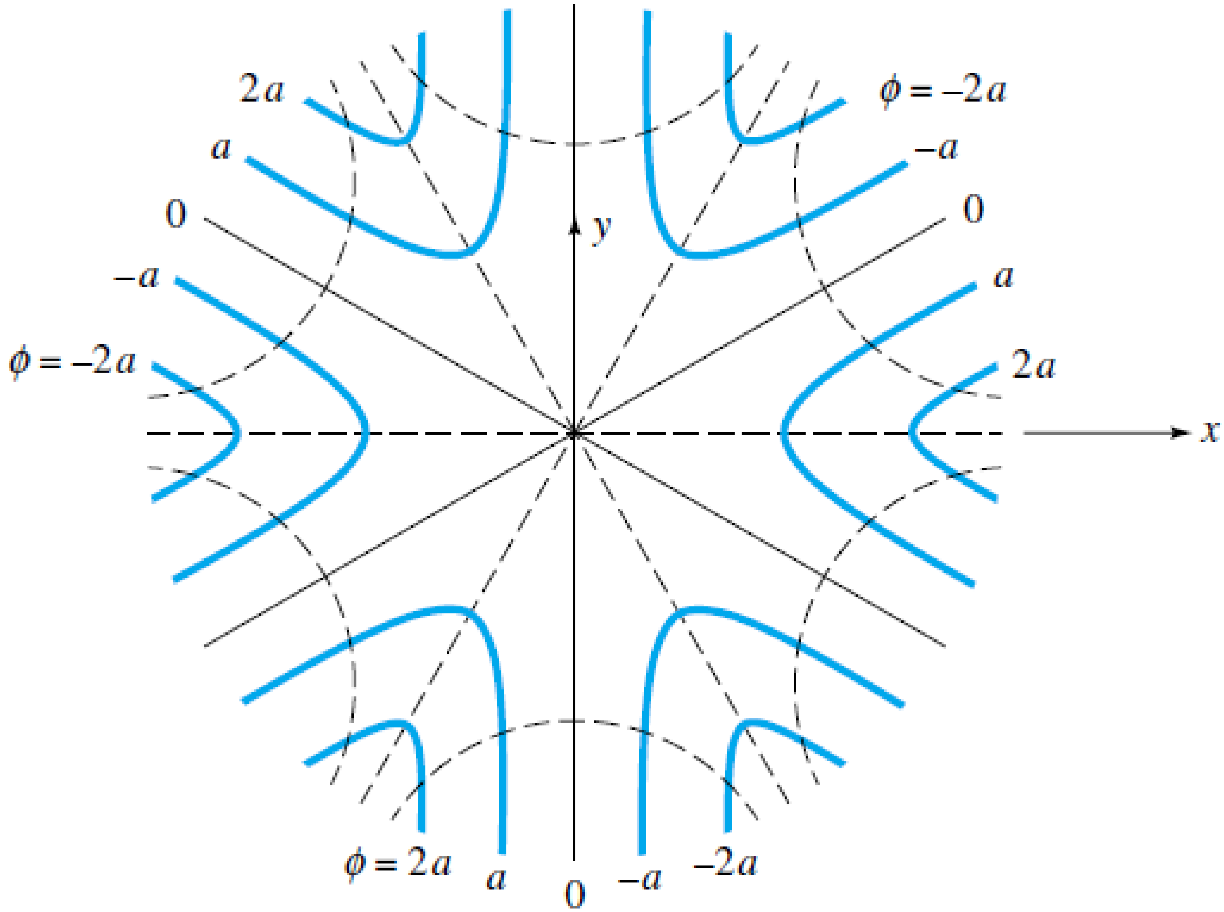
Velocity Potential

If a velocity potential exists for the velocity field

$$u = a(x^2 - y^2) \quad v = -2axy \quad w = 0$$

find it, plot it, and interpret it.

Velocity Potential



We expected trouble at the stagnation point, and there is no general rule for determining the behavior of the lines at that point.

Plane Polar Coordinates

Many solutions are conveniently expressed in polar coordinates (r, θ). Both the velocity components and the differential relations for ψ and ϕ are then changed, as follows:

$$v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Laplace's equation takes the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

Exactly the same equation holds for the polar-coordinate form of ψ (r, θ).