Integral calculus

Michael Stich

December 19, 2020

9 Integration of functions

Integral calculus has a tight relationship with differential calculus. Its essence is represented by the fundamental theorem of calculus that describes integration as inverse process to differentiation. Nevertheless, to understand better that relationship, we consider first the integration of functions in its own right. The historical motivation and fundamental application of integration is to calculate volumes, areas limited by curves and functions, and the length of curves. Generalizations of the methods involved lead to the calculation of work, velocity, moments of inertia etc. and other applications in science and engineering.

9.1 Definite integral of a function

The specific question that we will try to answer here is to calculate the area between the curve of a function $f(x)$ and the abscissa (the x-axis) in an interval $[a, b]$. To ensure that the area is finite, we assume that the function and the interval are bound. A closed and bound interval is called compact.

We define the partition P_n of the interval $[a, b]$ in $n + 1$ points:

$$P_n = \{x_0, x_1, x_2, ..., x_n\} \quad \text{with } a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

It is not necessary to assume that the points are equidistant, i.e., that the n subintervals with lengths $x_i - x_{i-1} = \Delta x_i$ (with $i = 1, 2, \cdots, n$) are all of the same length. Now, we define a set T_n of n intermediate points, with

$$T_n = \{x_1^*, x_2^*, ..., x_n^*\} \quad \text{with } x_{i-1} \le x_i^* \le x_i, \quad i = 1, 2, \cdots, n$$

and define the *Riemann sum* as

$$S_n = \sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}).$$

Every term of the sum is the product of the value of the function with an interval of x and hence represents the area of a rectangle with sides $f(x_i^*)$ and $x_i - x_{i-1}$ (compare with Fig. 1).

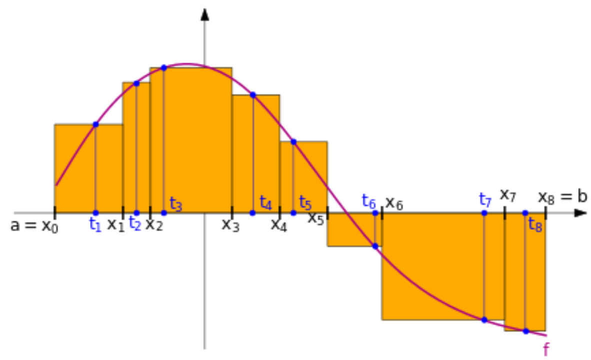

Figure 1: Construction of a Riemann sum for a función. The t_i represent the x_i^* of the text. Source: https://upload.wikimedia.org/wikipedia/commons/8/85/Riemannsumme.svg (public domain).

In general, the Riemann sum does *not* correspond exactly to the area between the curve and the abscissa, but only represents an approximation: by construction, the Riemann sum does not only depend on f but also on the specific partition and the choice of the intermediate points, both at the same time dependent on n: $S_n = S_n(f, P_n, T_n)$. However, the area below the curve must have a *unique* value and for that reason we eliminate the dependence on P_n, T_n and n by taking the limit $n \to \infty$, asking for the result not to depend on neither the partition nor the intermediate points:

Definition 1 (Riemann integral):
Let f be a function defined on the compact interval $[a, b]$ and P_n a partition such that the maximum length of the subintervals tends to zero when $n \to \infty$. If for all T_n

$$\lim_{n\to\infty} \left[\sum_{i=1}^{n} f(x_i^*)(x_i - x_{i-1}) \right] = I$$

exists, then

$$I = \int_a^b f(x)dx$$

is the Riemann integral of f on $[a, b]$.

If the function f is bound in $[a, b]$ and has at most a finite number of discontinuities in $[a, b]$, the Riemann integral exists.

Comments:
(1) The Riemann integral is a *definite* integral, i.e., represents a real and specific number, a scalar. Later, we introduce integral functions called *antiderivatives* or *primitives*.

(2) We know (Chapter on Continuity) that a continuous function on a compact interval is bound and fulfills the hypotheses of Definition 1. More on this in Theorem 2.

(3) In many texts, the hypothesis is used that the function f is continuous, but this is not necessary since it is possible to prove that there are discontinuous functions for which a Riemann integral exists. Furthermore, it is possible to generalize the integral concept for non-bound functions and non-bound intervals (improper integrals).

(4) Other formulations of the Riemann integral use the definition of lower and upper sums:

$$L_P(f) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \quad \text{being } m_i \text{ the minimum of } f \text{ in the subinterval } i,$$
$$U_P(f) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \quad \text{being } M_i \text{ the maximum of } f \text{ in the subinterval } i.$$

If the minimum of all upper sums (choosing among all possible partitions) coincides with the maximum of all lower sums (choosing among all possible partitions), we have (also assuming that the maximum of all subintervals tends to zero)

$$\min_P U_p(f) = \max_P L_p(f) = \int_a^b f(x)dx.$$

This method (also known as the Darboux integral) is illustrated in Fig. 2.

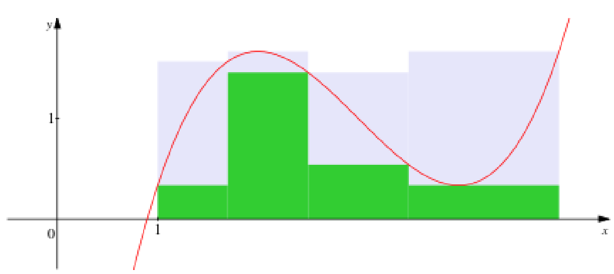

Figure 2: Lower and upper sums for a function. The area below the curve is bound to below by $L_P(f)$ and to above by $U_P(f)$. Source: https://upload.wikimedia.org/wikipedia/commons/5/59/Darboux.svg (dominio público).

(5) If the Riemann integral exists, then we can say that the function is Riemann integrable. There exist other integrability concepts (e.g. Lebesque), but unless stated

otherwise, integrability refers to Def. 1 (or, below, to the existence of an antiderivative).

(6) From the definition it becomes clear that we need to ensure a correct use of limits in a quite involved form (partitions, intermediate points). It is possible to make a comparison of the terms:

$$\lim_{n\to\infty}\left[\sum_{i=1}^{n} f(x_i^*)\Delta x_i\right] = \int_a^b f(x)dx.$$

The integral symbol $\int_a^b$ represents an infinite sum that covers the whole interval. The function part $f(x_i^*)$ is represented by $f(x)$ and the size Δx_i of the interval i, is represented by dx, el *differential*, standing for an infinitesimal increment, in complete analogy with dx in differential calculus.

(7) By convention, the areas are measured by positive numbers. To ensure that the integral returns a positive number, we ask for $f(x) > 0$ for all $x \in [a, b]$ without loss of generality. Below, we consider the cases when these conditions are not met.

Example 1:
Calculate the area below the function $f(x) = kx$, $k > 0$ in the interval $a \le b$ (with $a > 0$) using a Riemann integral.

For $k > 0$ and $a \le b$ (with $a > 0$), the area between the curve of the function and the abscissa lies in the first quadrant and represents a right trapezoid limited by $y = 0$, $y = kx$, $x = a$ and $x = b$. Since $f(x) > 0$ between a and b, the integral $I = \int_a^b kx\,dx$ indeed corresponds to this area.

We define a partition of $[a, b]$ in $n + 1$ points, choosing an equidistant partition with $\Delta x = (b - a)/n$ and setting:

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \cdots,\ x_n = a + n\Delta x = b,$$

with the "intermediate" points

$$x_1^* = a, x_2^* = a + \Delta x, x_3^* = a + 2\Delta x, \cdots, x_n^* = a + (n-1)\Delta x,$$

i.e., the intermediate points of each subinterval actually coincide with the lower limit of each subinterval (permitted according to the definition). The Riemann sum is

$$\begin{aligned} S_n &= \sum_{i=1}^{n} f(x_i^*)\Delta x_i = \sum_{i=1}^{n} f(x_i^*)\Delta x = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i^*) \\ &= \frac{b-a}{n}\left[ka + k(a+\Delta x) + k(a+2\Delta x) + \cdots + k(a+(n-1)\Delta x)\right], \end{aligned}$$

with n terms in the sum. Then:

$$\begin{aligned} S_n &= \frac{b-a}{n} \cdot k \left[a + (a+\Delta x) + (a+2\Delta x) + \cdots + (a+(n-1)\Delta x) \right] \\ &= \frac{b-a}{n} \cdot k \left[na + \Delta x (1+2+\cdots+(n-1)) \right]. \end{aligned}$$

We calculate the sum of the arithmetic progression $1+2+\cdots+(n-1) = n(n-1)/2$, replace Δx, and obtain

$$\begin{aligned} S_n &= \frac{b-a}{n} \cdot k \left[na + \frac{b-a}{n} \cdot \frac{n(n-1)}{2} \right] = \frac{b-a}{n} \cdot k \cdot n \left[a + \frac{b-a}{n} \cdot \frac{n-1}{2} \right] \\ &= k(b-a) \left[a + \frac{b-a}{2} \cdot \frac{n-1}{n} \right]. \end{aligned}$$

Now, we take the limit $n \to \infty$:

$$I = \lim_{n\to\infty} S_n = k(b-a) \left[a + \frac{b-a}{2} \right] = (b-a) \frac{ka+kb}{2},$$

coinciding with the area that can be obtained with elementary geometry (the product of the base of a trapezoid and the average length of the parallel sides).

We see that calculating integrals using the Riemann integral can be a tedious process even for relatively simple functions. Below, we see a more efficient method.

9.2 Basic properties of the integral

Theorem 1 (immediate properties):
Let f be a Riemann integrable function on an interval $[a, b]$. Then:
(i) Inversion of the interval: $\int_a^b f(x)dx = -\int_b^a f(x)dx$.
(ii) Zero interval: $\int_c^c f(x)dx = 0$ for all $c \in [a, b]$.
(iii) Independence of the integral on the variable of integration:
$I = \int_a^b f(x)dx = \int_a^b f(t)dt$.

Comment:
Part (iii) emphasizes that the argument of the function (the *integrand*) over which is integrated has to coincide with the variable of integration as identified through the differential (here, dx or dt). The result is a scalar number that does not depend on x (or t).

In Definition 1, it was stated that a bound function with at most a finite number of discontinuities is integrable. Here, we specify other types of integrable functions.

Theorem 2 (integrable functions):
(i) If f is continuous in $[a, b]$ compact, f is integrable in $[a, b]$.

(ii) If f is monotonic in $[a, b]$ compact, f is integrable in $[a, b]$.

Comments:
(1) A continuous function on a compact interval is bound and has no discontinuities and hence is integrable according to Definition 1.
(2) Assuming appropriate hypotheses about the intervals, we can establish that a continuous function is always integrable but it is not necessary since there are non-continuous functions which are integrable. Recalling that all derivable functions are continuous, we find that the class of integrable functions is larger than the class of continuous functions and the latter one larger than the class of differentiable functions.

Theorem 3 (basic properties):
Let f and g be integrable functions in $[a, b]$ compact. Then,
(i) Multiplication with a number: $\int_a^b c \cdot f(x)\,dx = c\int_a^b f(x)\,dx$, with $c \in \mathbb{R}$.

(ii) Property of the sum: $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(iii) Additivity with respect to the interval: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$ (if f is integrable in $[a, b]$ and $[b, c]$).

(iv) Monotonicity of the integral: If $f(x) \leq g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

(v) Integrability of the absolute value: If $f(x)$ is integrable in the interval $[a, b]$ with $a < b$, $|f(x)|$ is also integrable, with $|\int_a^b f(x)\,dx| \leq \int_a^b |f(x)|dx$.

(vi) Integrability of the product and the quotient: $f(x) \cdot g(x)$ and $f(x)/g(x)$ are integrable, the latter if for all $x \in [a, b]$ exists a $k > 0$ such that $|g(x)| \geq k$.

9.3 The fundamental theorem of calculus

Like in many contexts of mathematics, to formulate certain properties and results require the previous definition of appropriate concepts. To relate integral calculus with differential calculus we need to move from the definite integral (a number) to the integral function (in analogy to the derivative in a point and the derivative as function). First, we need to define for a function that function whose derivative is the original function:

Definition 2 (antiderivative):
A function $f : [a, b] \to \mathbb{R}$ has an antiderivative (or primitive) ϕ if ϕ is differentiable with $\dfrac{d}{dx}\phi(x) = f(x)$ for all $x \in [a, b]$.

This definition does not provide any method to calculate the antiderivative for a given function f, neither gives a criterion on f to ensure the existence of ϕ. It is a mere definition of the relation between f and ϕ. To proceed, we need to generalize the definite integral and define the integral function.

Theorem 4 (integral function):
If the function f is integrable on $[a, b]$ compact, then the function $\phi_a(x) = \int_a^x f(t)\, dt$, with $x \le b$ exists and is continuous in $[a, b]$.

In the first place, we have defined an integral as function of the upper limit of the interval. The independent variable of ϕ is x, the upper limit of the interval. For that reason, the variable of integration has to be identified with another letter, here t. It is an error to use the same letter in the same equation for both concepts.

We observe that the function $\phi_a(x)$ is *continuous* according to Theorem 4. So while the original function need not to be continuous, its integral is! However, we could wish for more: If the integral function was differentiable, we could take its derivative and compare it to the original function f. It is the fundamental theorem of calculus that precisely establishes that relationship! We see that the integrability of f is not sufficient, we have to ask for f being continuous.

Theorem 5 (fundamental theorem of calculus, FTC):
If the function f is continuous in $[a, b]$ compact, then the function $\phi_a(x) = \int_a^x f(t)\, dt$, with $x \le b$ exists and is differentiable in $[a, b]$ and its derivative is $\dfrac{d}{dx}\phi_a(x) = f(x)$.

Comments:
(1) This implies that $\phi_a(x)$ is an antiderivative of $f(x)$, according to Definition 2.

(2) The derivative can be written as

$$\frac{d}{dx}\left(\int_a^x f(t)\, dt\right) = f(x),$$

showing that integration and differentiation are inverse processes.

Theorem 6 (set of antiderivatives):
If the function f is continuous in $[a, b]$ compact, then f has a set of antiderivatives in $[a, b]$ with $\phi(x) = \int_a^x f(t)\, dt + C$, with $C \in \mathbb{R}$.

Proof:
$\phi(x)$ is an antiderivative according to Def. 2 if it is differentiable and its derivative is

identical to function f. We calculate

$$\begin{aligned}\frac{d}{dx}(\phi(x)) &= \frac{d}{dx}\left(\int_a^x f(t)\,dt + C\right) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) + \frac{d}{dx}C \\ &= \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) + 0 = f(x)\end{aligned}$$

The meaning of Theorem 6 is that we can add a constant to any antiderivative and it remains an antiderivative. This leads us to the concept of the indefinite integral:

Definition 3 (indefinite integral):
The set of all antiderivatives of a function f is called indefinite integral and we write:

$$\int f(x)\,dx = \phi(x) + C, \quad C \in \mathbb{R} \quad \text{if and only if} \quad \phi'(x) = f(x).$$

Comment:
The indefinite integral relates two functions with the same independent variable x and in absence of any interval limit the integration variable is also x.

How can we use the antiderivatives to calculate definite integrals? The following theorem provides a specific rule.

Theorem 7 (Barrow's rule):
Let the function f be continuous in $[a, b]$ compact and ϕ any antiderivative of f (in this interval), then

$$I = \int_a^b f(x)\,dx = \phi(b) - \phi(a) = \phi(x)|_{x=a}^{x=b} = [\phi(x)]_a^b.$$

With this rule, it is possible to calculate the definite integral not via a process of sums and limits like in Riemann's integral, but as a subtraction of values of an auxiliar function evaluated at the interval boundaries. That auxiliar function is precisely the antiderivative. In the next section we study how to calculate the antiderivative.

9.4 Integration techniques

To solve integration problemas (both definite and indefinite integrals), it is common to find the antiderivative first. For this task there is a range of strategies:

- Invert the rules of differentiation (immediate and almost immediate integrals)
- Simplify the function using Theorems 1 and 3
- Partial integration (see below)

- Integration through substitution (see below)
- Rule of logarithmic integration (see below)
- Use selected methods for special function types, exemplified below for rational functions and trigonometric functions

If necessary, use then Barrow's rule to determine the definite integral.
Recall: Differentiation is a craft, integration is an art!

9.4.1 Partial integration

Let us recall the product rule of differentiation:
Let f and g be differentiable functions. Then, $(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$. A way of interpreting this rule is to identify $f(x)g(x)$ as the antiderivative of $f'(x)g(x) + f(x)g'(x)$. Then, we can integrate the equation and we obtain

$$\begin{aligned}\int (f'(x)g(x) + f(x)g'(x))\,dx &= f(x)g(x) + C,\\ \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx &= f(x)g(x) + C.\end{aligned}$$

Recasting the terms we obtain as rule for partial integration the following theorem.

Theorem 8 (partial integration):
Let f and g be differentiable functions (with their derivatives being continuous) over $[a,b]$ compact. Then,

$$\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx + C.$$

As always in the context of the indefinite integral, we have to carry a constant of integration: any constant may be added to an antiderivative since the derivative of any the antiderivatives yields the same function.

For definite integrals, we obtain

$$\int_a^b f'(x)g(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f(x)g'(x)\,dx.$$

The rule replaces one integral (on the left-hand side) by two terms, one is the product of functions evaluated in the limits of the interval and another integral (on the right-hand side), still unsolved, justifying the name of partial integration. This rule is used if the integral on the right-hand side is easier to determine than the one on the left-hand side. By construction, this rule can be interpreted as the inversion of the product rule of differentiation.

Example 2: Calculate the following indefinite integrals using partial integration:

(a) $F(x) = \int xe^x \, dx$.
We know how to integrate polynomials and exponential functions, but we do not know the integral of this product. We identify x with $g(x)$ and e^x with $f'(x)$. It is easy to integrate $f'(x)$ and we obtain $f(x) = e^x$, at the same time as $g'(x) = 1$. Therefore,

$$\int xe^x \, dx = xe^x - \int 1 \cdot e^x \, dx + C = xe^x - e^x + C = e^x(x-1) + C.$$

(b) $F(x) = \int x^2 \sin x \, dx$.
We identify x^2 with $g(x)$ and $\sin x$ with $f'(x)$. It is easy to integrate $f'(x)$ and we obtain $f(x) = -\cos x$, at the same time as $g'(x) = 2x$. Therefore,

$$\int x^2 \sin x \, dx = x^2(-\cos x) + \int (2x) \cdot \cos x \, dx + C.$$

The integrand has lowered the order in the polynomial part and we have to apply partial integration again to this integral, identifying $2x$ with $g(x)$ ($g'(x) = 2$) and $\cos x$ with $f'(x)$ ($f(x) = \sin x$):

$$F(x) = -x^2 \cos x + \left(2x \sin x - 2 \int \sin x \, dx\right) + C.$$

In principle, the term $2x \sin x$ provides a new constant of integration, but we can absorb the two constants of integration in one (the sum of two arbitrary numbers is just another arbitrary number). We integrate again and obtain the final result:

$$F(x) = -x^2 \cos x + 2x \sin x - 2(-\cos x) + C = (2 - x^2) \cos x + 2x \sin x + C.$$

(c) $F(x) = \int \ln x \, dx$.
Although the integrand is not a product, we can interpret it as such by introducing a factor 1, identifying $f'(x) = 1$ ($f(x) = x$) and $g(x) = \ln x$ ($g'(x) = \dfrac{1}{x}$). We obtain:

$$\begin{aligned} \int \ln x \, dx &= x \cdot \ln x - \int x \cdot \frac{1}{x} \, dx + C \\ &= x \cdot \ln x - \int 1 \, dx + C = x \cdot \ln x - x + C, \end{aligned}$$

where we have absorbed the constants of integration in one.

(d) $F(x) = \int \frac{1}{x} \, dx$.
We know that $(\ln x)' = \dfrac{1}{x}$ and therefore we can solve directly

$$\int \frac{1}{x} \, dx = \ln x + C,$$

with the constant of integration. Nevertheless, we try to apply partial integration, identifying $f'(x) = 1$ ($f(x) = x$) and $g(x) = \frac{1}{x}$ ($g'(x) = -\frac{1}{x^2}$). We obtain

$$\begin{aligned} \int \frac{1}{x}\,dx &= x \cdot \frac{1}{x} - \int x \cdot \frac{-1}{x^2}\,dx + C \\ &= 1 + \int x \cdot \frac{1}{x^2}\,dx + C = 1 + \int \frac{1}{x}\,dx + C. \end{aligned}$$

On the right-hand side we reproduce the initial integral, plus $1 + C$. This result shows that partial integration in this example does not lead us to the primitive. But it tells us how to interpret the result and the constant of integration. With C being an arbitrary constant, $C + 1$ also is, and we can rename it as C and write

$$\int \frac{1}{x}\,dx = \int \frac{1}{x}\,dx + C.$$

This is not an incorrect result because both $\int \frac{1}{x}\,dx$ and $\int \frac{1}{x}\,dx + C$ are antiderivatives of $\frac{1}{x}$ and we have to recall that, in spite of the result, we have integrated once and therefore there must be a constant of integration. If we calculate a definite integral in this example (with $0 < a < b$), no constant of integration appears:

$$\int_a^b \frac{1}{x}\,dx = \left[x \cdot \frac{1}{x}\right]_a^b - \int_a^b x \cdot \frac{-1}{x^2}\,dx = (1-1) + \int_a^b x \cdot \frac{1}{x^2}\,dx = \int_a^b \frac{1}{x}\,dx.$$

The strategy to introduce a factor 1 which is identified with $f'(x)$ is valid and can be useful to solve other integrals.

The mnemotecnic rule ALPES is a way to memorize the order in which the functions are normally assigned: Arcsine (and other inverse trigonometric functions), Logarithm, Polynomial, Exponential, Sine (and other trigonometric functions): the type of function that appears first is the one assigned to $g(x)$, el second one to $f'(x)$. This rule works because the functions easier to integrate are in the lower part of the list.

Applying partial integration, one tends to encounter one of the following cases:

- the rule solves the integral directly (Ex. 2(c));
- the rule simplifies the integral, but the process has to be repeated (e.g. to lower the order of the polynomial until a constant is reached) to solve the integral (Ex. 2(b));
- the rule does not give a useful result (Ex. 2(d));
- the rule produces a term which is identical to the original integral, but with inverted sign, that is: $F(x) = h(x) - F(x) + C$, where $F(x)$ represents the integral to solve and $h(x)$ a non-constant function. Then, $2F(x) = h(x) + C$ and we solve $F(x) = h(x)/2 + C$.

9.4.2 Integration through substitution

This method is also known as integration via change of variables and can be interpreted as the inversion of the chain rule of differentiation.

Theorem 9 (integration through substitution): Let f be a continuous function in an interval J and $g(x)$ a differentiable function with continuous derivative in a compact interval $[a, b]$ and image J. Then,

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)dt.$$

Comments:
(1) $t = g(x)$ represents the change of variable from x to t.

(2) The typical use of Theorem 9 is to replace the integral on the left-hand side (representing the product of two functions) by the integral on the right-hand side and proceed to resolve the right-hand side with other means (e.g., as immediate integral). For that, it is important to recognize that one of the factors $(g'(x))$ is the derivative of the argument of the other factor $(f(g(x)))$ (see example below). Nevertheless, Theorem 9 works in both ways, i.e., also from right to left if appropriate.

(3) While performing a change of variables, the variable of integration and the limits of integration also change. The change of variable of integration is visible in the differential dt: it would be wrong to integrate the right side over x.

(4) For an interval $[a, b]$, we know that $a < b$ (for $a = b$, the integral would be zero), but that does not imply that after the change of variable $g(a) < g(b)$ is fulfilled. Indeed, the simple change $t = g(x) = -x$ implies that $g(b) < g(a)$ and in the integral $\int_{g(a)}^{g(b)} f(t)dt$ the lower limit is larger than the upper limit. As usual, we put the lower value as lower limit and use (compare with Theorem 1(i)):

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \begin{cases} \int_{g(a)}^{g(b)} f(t)dt & \text{si } g(a) < g(b), \\ -\int_{g(b)}^{g(a)} f(t)dt & \text{si } g(b) < g(a). \end{cases}$$

(5) For indefinite integrals, the rule is formulated as

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(t)\, dt,$$

without any constant of integration because only one integral has been replaced by another one. Resolving the integral $\int f(t)\, dt$, the result is a function that depends on t, while a result depending on x is expected. To complete the calculation, we have to resubstitute the result using

$$x = g^{-1}(t),$$

assuming that the inverse function g^{-1} of the change of variable is well-defined.

(6) While changing the variable of integration, the differential is changed:

$$\begin{aligned} t &= g(x), \\ \frac{dt}{dx} &= g'(x), \\ dt &= g'(x)\,dx. \end{aligned}$$

While implementing a change of variable $t = g(x)$ in the integral $\int f(t)\,dt$ one has to replace $f(t)$ by $f(g(x))$ and dt by $g'(x)\,dx$ as the theorem states.

(7) The integration by substitution can be interpreted as the inversion of the chain rule for differentiation since the derivative of a composed function is given by $(f(g(x)))' = f'(g(x)) \cdot g'(x)$.

Example 3: Calculate the following integrals using a change of variable:

(a) $I = \int_0^2 x \cos(x^2+1)\,dx$.
We observe that the derivative of the argument of the cosine is $2x$ which is (disregarding a constant factor) the other factor of the function. We substitute

$$t = g(x) = x^2 + 1$$

and calculate $dt/dx = g'(x) = 2x$ and therefore $x\,dx = dt/2$. We change the intervals:

$$\begin{aligned} x = 0 \;&\widehat{=}\; t = 1, \\ x = 2 \;&\widehat{=}\; t = 5. \end{aligned}$$

We calculate

$$I = \int_0^2 x\cos(x^2+1)\,dx = \int_1^5 \cos t \cdot (\frac{1}{2}\,dt) = \frac{1}{2}\int_1^5 \cos t\,dt = \frac{1}{2}[\sin t]_1^5 = \frac{1}{2}(\sin 5 - \sin 1).$$

(b) $F(x) = \int x \cos(x^2+1)\,dx$.
We use the same change of variable, but there are no interval limits to be changed. But a constant of integration has to be included and t must be replaced by x after integration. In many cases it is helpful to manipulate the expression to recognize $g(x)$ and $g'(x)$ to prepare the substitution:

$$\begin{aligned} F(x) &= \int x\cos(x^2+1)\,dx = \frac{1}{2}\int \cos(x^2+1)\cdot(2x)\,dx = \frac{1}{2}\int \cos t\,dt \\ &= \frac{1}{2}\sin t + C = \frac{1}{2}\sin(x^2+1) + C. \end{aligned}$$

(c) $I = \int_0^r \sqrt{r^2 - x^2}dx$, con $r \in \mathbb{R}^+$.
In this example we perform the change of variable in the other direction (from one

term to two). The reason is that the function describes one part of a circle of radius r and this suggests a change of variable to angle t. We substitute and calculate:

$$\begin{aligned} x &= g(t) = r\sin t, \\ \frac{dx}{dt} &= g'(t) = r\cos t, \\ dx &= r\cos t\,dt, \\ x = 0 &\mathrel{\widehat{=}} t = 0, \\ x = r &\mathrel{\widehat{=}} t = \frac{\pi}{2}, \end{aligned}$$

where $x = g(t)$ (names of the variables are arbitrary).

$$I = \int_0^r \sqrt{r^2 - x^2}dx = \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t} \cdot (r\cos t)\,dt,$$

where the root represents $f(g(t))$ and $r\cos t$ the part $g'(t)$. The justification is in the next step in which we use the trigonometric property $\sin^2 u + \cos^2 u = 1$:

$$I = r^2 \int_0^{\pi/2} \sqrt{1 - \sin^2 t} \cdot \cos t\,dt = r^2 \int_0^{\pi/2} \cos^2 t\,dt.$$

In the next step, we can perform a partial integration with $g'(t) = \cos t$ and $f(t) = \cos t$ or use the trigonometric property $2\cos^2 u = 1 + \cos(2u)$:

$$\begin{aligned} I &= r^2 \int_0^{\pi/2} \left(\frac{1}{2} + \frac{1}{2}\cos(2t)\right) dt = \frac{r^2}{2}\int_0^{\pi/2} dt + \frac{r^2}{2}\int_0^{\pi/2} \cos(2t)\,dt \\ &= \frac{r^2}{2}[t]_0^{\pi/2} + \frac{r^2}{2}\int_0^{\pi/2} \cos(2t)\,dt \\ &= \frac{\pi r^2}{4} + \frac{r^2}{2}\int_0^{\pi/2} \cos(2t)\,dt. \end{aligned}$$

Now, we replace

$$\begin{aligned} t &= h(p) = p/2, \\ \frac{dt}{dp} &= h'(p) = 1/2, \\ dt &= dp/2, \\ t = 0 &\mathrel{\widehat{=}} p = 0, \\ t = \frac{\pi}{2} &\mathrel{\widehat{=}} p = \pi, \end{aligned}$$

and obtain

$$I = \frac{\pi r^2}{4} + \frac{r^2}{2}\int_0^{\pi} \cos(p)\frac{1}{2}\,dp = \frac{\pi r^2}{4} + \frac{r^2}{4}[\sin p]_0^{\pi} = \frac{\pi r^2}{4} + \frac{r^2}{4}(0 - 0) = \frac{\pi r^2}{4}.$$

(d) $F(x) = \int \sqrt{r^2 - x^2} dx$, with $r \in \mathbb{R}^+$ y $|x| \leq r$.
We use part of (c), to be specific the two changes of variable and we write:

$$F(x) = \frac{r^2}{2} t(x) + \frac{r^2}{4} \sin p(x) + C,$$

where $t(x)$ and $p(x)$ indicate that we have to substitute t and p by x:

$$F(x) = \frac{r^2}{2} \left(t(x) + \frac{1}{2} \sin(2t(x)) \right) + C,$$

using the trigonometric identity $\sin(2u) = 2 \sin u \cos u$:

$$F(x) = \frac{r^2}{2} \left(t(x) + \sin t(x) \cdot \cos t(x) \right) + C.$$

Since $x = r \sin t$, we have $t = \arcsin(\frac{x}{r})$ and

$$\begin{aligned} F(x) &= \frac{r^2}{2} \left(\arcsin(\frac{x}{r}) + \sin(\arcsin(\frac{x}{r})) \cdot \cos(\arcsin(\frac{x}{r})) \right) + C, \\ &= \frac{r^2}{2} \left(\arcsin(\frac{x}{r}) + \frac{x}{r} \cdot \cos(\arcsin(\frac{x}{r})) \right) + C. \end{aligned}$$

Now, we apply another trigonometric identity, $\cos(\arcsin(u)) = \sqrt{1 - u^2}$,

$$F(x) = \frac{r^2}{2} \left(\arcsin(\frac{x}{r}) + \frac{x}{r} \cdot \sqrt{1 - \frac{x^2}{r^2}} \right) + C,$$

representing the final result. One can confirm with Barrow's rule that $F(r) - F(0) = \frac{\pi r^2}{4}$, the result of (c).

9.4.3 Rule of logarithmic integration

Theorem 10 (rule of logarithmic integration): Let g be a function with $g(x) \neq 0$ and differentiable with $g'(x)$ continuous. Then,

$$\int \frac{g'(x)}{g(x)} \, dx = \ln |g(x)| + C.$$

Comment:
This rule can be interpreted as the inversion of the chain rule of the derivative of the logarithm of a positive function, $\frac{d}{dx}(\ln |g(x)|) = g'(x)/g(x)$.

Example 4: Solve the following integrals:

(a) $\int \frac{1}{x \ln x}\, dx$.

If we fix $g(x) = \ln x$, we know that $g'(x) = 1/x$ and hence

$$\int \frac{1}{x \ln x}\, dx = \int \frac{\frac{1}{x}}{\ln x}\, dx = \int \frac{g'(x)}{g(x)}\, dx = \ln |g(x)| + C = \ln |\ln x| + C.$$

(b) $\int \frac{dx}{\sqrt{1-x^2}\arcsin x}$.

To apply the rule, la function need not to contain logarithmic functions. We recognize the derivative of the arcsine $1/\sqrt{1-x^2}$. Therefore, we choose $g(x) = \arcsin x$ and $g'(x) = 1/\sqrt{1-x^2}$. Therefore,

$$\int \frac{dx}{\sqrt{1-x^2}\arcsin x} = \int \frac{g'(x)}{g(x)}\, dx = \ln |g(x)| + C = \ln |\arcsin x| + C.$$

9.4.4 Integration of rational functions

A rational function $R(x)$ is the quotient of two polynomials $P(x)$ and $Q(x)$:

$$R(x) = \frac{P(x)}{Q(x)} = \frac{a_u x^u + a_{u-1}x^{u-1} + \cdots + a_0}{b_v x^v + b_{v-1}x^{v-1} + \cdots + b_0},$$

where u and v are the degrees of the two polynomials, respectively. Before we show how to determine integrals of the kind

$$\int R(x)dx = \int \frac{P(x)}{Q(x)}dx,$$

we anticipate that it is *always* possible to give a solution and that it will be composed of a linear combination of rational functions, logarithms of linear functions, logarithms of quadratic functions, and arctangents of linear functions.

The first case to consider is when $u \geq v$. Then, we perform a division of polynomials (e.g., Ruffini's rule) so that we obtain

$$R(x) = H(x) + \frac{P_1(x)}{Q(x)}.$$

Now, $H(x)$ stands for a polynomial that we integrate with the standard rule of integration of polynomials and we only need to add to this the integral of $\frac{P_1(x)}{Q(x)}$ where the degree of $P_1(x)$ now is strictly smaller than the degree of $Q(x)$.

The rational function $R(x)$ admits a *partial fractions decomposition* and for that we use that any polynomial (here, $Q(x)$) can be factorized using its roots:

$$Q(x) = c(x-r_1)^{n_1}(x-r_2)^{n_2}\cdots(x-r_s)^{n_s}[(x-p_1)^2+q_1^2]^{m_1}[(x-p_2)^2+q_2^2]^{m_2}\cdots[(x-p_t)^2+q_t^2]^{m_t},$$

where $r_1, r_2, \ldots, r_s$ are s different real roots of $Q(x)$ (with multiplicities $n_1, n_2, \ldots, n_s$) and $p_1 \pm iq_1, p_2 \pm iq_2, \ldots$ are t different complex roots of $Q(x)$ (with multiplicities $m_1, m_2, \ldots, m_t$) and c is a constant. Simple roots with multiplicities 1 are included in this notation.

For each root (corresponding to a factor of $Q(x)$) we can use the following table to determine the term(s) that have to appear in the partial fraction decomposition (PFD).

Root factor of $Q(x)$	Term in PFD
$(x-r)^n$	$\sum_{i=1}^{n} \frac{K_i}{(x-r)^i}$
$[(x-p)^2+q^2]^m$	$\sum_{i=1}^{m} \frac{M_i x + N_i}{[(x-p)^2+q^2]^i}$

This includes simple roots (with multiplicities 1).

In this way, the partial fractions decomposition of a rational function is then schematically given by a summation of terms,

$$\frac{P_1(x)}{Q(x)} = \frac{1}{c}\left[\sum_{k=1}^{s}\sum_{i=1}^{n_k}\frac{K_{i,k}}{(x-r_k)^i} + \sum_{k=1}^{t}\sum_{i=1}^{m_k}\frac{M_{i,k}x + N_{i,k}}{[(x-p_k)^2+q_k^2]^i}\right],$$

where the first group of sums refer to the real roots and the second group to the complex roots. The unknown coefficients K, M, N in the numerators of the right-hand side are determined via comparison with the left-hand side when both sides are multiplied with $Q(x)$ and terms of same order are compared. Note that the left-hand side is then $P_1(x)$. This task and the partial fractions decomposition in general are part of basic algebra and not related to integration.

We emphasize that there are only 4 types of fractions that appear in the decomposition (real or complex roots, with or without multiplicity). Since the integral of a sum is the sum of the integrals, we only need to state the solutions for these 4 types of integrals (where K, M, N represent real numbers determined in the partial fractions decomposition):

$$\begin{aligned}
\int \frac{K}{x-r}dx &= K\ln|x-r| + C, \\
\int \frac{K}{(x-r)^n}dx &= \frac{-K}{(n-1)(x-r)^{n-1}} + C \qquad \text{with } n > 1, \\
\int \frac{Mx+N}{(x-p)^2+q^2}dx &= \frac{M}{2}\ln[(x-p)^2+q^2] + \frac{Mp+N}{q}\arctan\left(\frac{x-p}{q}\right) + C, \\
\int \frac{Mx+N}{[(x-p)^2+q^2]^m}dx &= \frac{M}{2(1-m)[(x-p)^2+q^2]^{m-1}} + (Mp+N)\cdot I_m(x), \quad m > 1,
\end{aligned}$$

where $I_m(x) = \int \frac{dx}{[(x-p)^2+q^2]^m}$ is determined recursively with

$$\begin{aligned} I_m(x) &= \frac{x-p}{2(m-1)q^2[(x-p)^2+q^2]^{m-1}} + \frac{2m-3}{2(m-1)q^2} I_{m-1}(x), \\ I_1(x) &= \frac{1}{q}\arctan\left(\frac{x-p}{q}\right) + C. \end{aligned}$$

Examples are provided in the work sheet.

9.4.5 Integration of trigonometric functions

We have already used a change of variable involving trigonometric functions in Example (3c,d). Here, we give a table of suitable substitutions for functions that involve either trigonometric functions or functions to which trigonometric identities can be applied. The following table is not exhaustive and does not give a single substitution that fits all cases. A lot of practice and trial-and-error is needed to solve integrals. The variable names are arbitrary.

If $f(x)$ contains	try	with
$\sqrt{a^2-x^2}$	$x = a\sin\theta$ or $x = a\tanh u$	$\frac{dx}{d\theta} = a\cos\theta$ $\frac{dx}{du} = a\,\mathrm{sech}^2 u$
$\sqrt{a^2+x^2}$	$x = a\sinh u$ or $x = a\tan\theta$	$\frac{dx}{du} = a\cosh u$ $\frac{dx}{d\theta} = a\sec^2\theta$
$\sqrt{x^2-a^2}$	$x = a\cosh u$ or $x = a\sec\theta$	$\frac{dx}{du} = a\sinh u$ $\frac{dx}{d\theta} = a\sec\theta\tan\theta$
Circular functions	$s = \sin x$ or $c = \cos x$ or $t = \tan(\frac{x}{2})$	$\frac{ds}{dx} = \cos x$ $\frac{dc}{dx} = -\sin x$ $\frac{dx}{dt} = \frac{2}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$
Hyperbolic functions	$u = e^x$ or $s = \sinh x$ or $c = \cosh x$ or $t = \tanh(\frac{x}{2})$	$\frac{du}{dx} = e^x$ $\frac{ds}{dx} = \cosh x$ $\frac{dc}{dx} = \sinh x$ $\frac{dt}{dx} = \frac{1}{2}\mathrm{sech}^2(\frac{x}{2})$

Of particular interest is the substitution $t = \tan(\frac{x}{2})$ because it allows to replaces all simple sine and cosine functions with rational functions.

Examples are provided in the work sheet.

9.5 Improper integrals

We recall that the Riemann integral was defined for a bound function f over a compact (bound and closed) interval $[a, b]$.

The definite integral $\int_a^b f(x)dx$ is improper if at least one of the following hypotheses are met: (a) The interval is not bound. (b) The function f is not bound in the interval.

9.5.1 Integrals in unbound intervals (improper integrals of the first kind)

Integrals in unbound intervals (also known as improper integrals of the first kind) can be found if either the lower bound of the interval in question is $-\infty$ or the upper bound of the interval is $+\infty$, or both. We therefore distinguish the following possibilities:
(a) Let f be bound and integrable in $[M, b]$, with $b \in \mathbb{R}$ and $M \leq b$. The improper integral is defined as

$$I = \int_{-\infty}^{b} f(x)dx = \lim_{M\to-\infty} \int_{M}^{b} f(x)dx.$$

(b) Let f be bound and integrable in $[a, N]$, with $a \in \mathbb{R}$ and $a \leq N$. The improper integral is defined as

$$I = \int_{a}^{\infty} f(x)dx = \lim_{N\to\infty} \int_{a}^{N} f(x)dx.$$

(c) Let f be bound and integrable in $[M, N]$, with $M \leq N$. The improper integral is defined as

$$I = \int_{-\infty}^{\infty} f(x)dx = \lim_{M\to-\infty} \int_{M}^{c} f(x)dx + \lim_{N\to\infty} \int_{c}^{N} f(x)dx,$$

with $c \in \mathbb{R}$ and $M < c < N$.

Obviously, the naming of the quantities M, N is arbitrary. We have defined the improper integral as the limit of a proper definite integral. There are three different outcomes of the limit process:
(i) If the limit does not exist but is also not infinite (in case (c), in at least one limit), the improper integral does not exist.
(ii) If the limit does not exist and is $+\infty$ or $-\infty$ (in case (c), in at least one limit and if the other limit is not covered by (i)), the improper integral is divergent with $I = +\infty$

or $I = -\infty$.
(iii) If the limit exists and is finite (in case (c), in both limits), the improper integral is convergent and its value is I.

Example 5:
Type (a)

$$\int_{-\infty}^{0} e^x dx = \lim_{M\to-\infty} \int_{M}^{0} e^x dx = \lim_{M\to-\infty} [e^x]_M^0 = \lim_{M\to-\infty} (1 - e^M) = 1 \quad \text{convergent.}$$

Type (b)

$$\int_{1}^{\infty} \frac{1}{x} dx = \lim_{N\to\infty} \int_{1}^{N} \frac{1}{x} dx = \lim_{N\to\infty} [\ln x]_1^N = \lim_{N\to\infty} \ln N = \infty \quad \text{divergent.}$$

Type (c)

$$I = \int_{-\infty}^{\infty} xe^{-x^2} dx.$$

First, let us calculate the primitive using a change of variable $t = x^2$:

$$\int xe^{-x^2} dx = \frac{1}{2} \int 2xe^{-x^2} dx = \frac{1}{2} \int e^{-t} dt = -\frac{1}{2} e^{-t} + C = -\frac{1}{2} e^{-x^2} + C$$

Therefore,

$$\int_{0}^{N} xe^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_0^N = -\frac{1}{2} e^{-N^2} + \frac{1}{2} = \frac{1}{2}(1 - e^{-N^2}),$$

and consequently

$$\int_{0}^{\infty} xe^{-x^2} dx = \lim_{N\to\infty} \left(\frac{1}{2}(1 - e^{-N^2}) \right) = \frac{1}{2}.$$

Similarly, for the other part of the improper integral:

$$\int_{M}^{0} xe^{-x^2} dx = \left[-\frac{1}{2} e^{-x^2} \right]_M^0 = -\frac{1}{2} + \frac{1}{2} e^{-M^2} = -\frac{1}{2}(1 - e^{-M^2}),$$

and consequently

$$\int_{-\infty}^{0} xe^{-x^2} dx = \lim_{M\to-\infty} \left(-\frac{1}{2}(1 - e^{-M^2}) \right) = -\frac{1}{2}.$$

In this case we find

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = -\frac{1}{2} + \frac{1}{2} = 0$$

and the improper integral is convergent with $I = 0$. In this example, we put $c = 0$ because $e^0 = 1$ and the calculation simplifies, but any other choice of c would also work. Also, the function xe^{-x^2} is actually an odd function and we could have used $\int_{-\infty}^{0} xe^{-x^2}dx = -\int_{0}^{\infty} xe^{-x^2}dx$.

A valid formulation in this case is also

$$I = \int_{-\infty}^{\infty} xe^{-x^2}dx = \lim_{M\to-\infty,\, N\to\infty} \left[-\frac{1}{2}e^{-x^2}\right]_M^N = \lim_{M\to-\infty,\, N\to\infty} \left[-\frac{1}{2}e^{-N^2} + \frac{1}{2}e^{-M^2}\right],$$

and from there

$$I = \lim_{M\to-\infty}\left(\frac{1}{2}e^{-M^2}\right) + \lim_{N\to\infty}\left(-\frac{1}{2}e^{-N^2}\right) = 0,$$

where we have used Barrow's rule and that the individual limits exist. In a further simplification, we could have put $\int_{-\infty}^{\infty} xe^{-x^2}dx = \lim_{K\to\infty}\int_{-K}^{K} xe^{-x^2}dx$.

9.5.2 Integrals of unbound functions (improper integrals of the second kind)

Improper integrals of the second kind can be found if the function tends to $+\infty$ or $-\infty$ at a finite value of x (which is either at the interval boundary or at an inner point). We therefore distinguish the following possibilities:
(a) Let f be integrable in $[a,b)$ and $\lim_{x\to b-}|f(x)| = \infty$. The improper integral is defined as

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0+}\int_a^{b-\epsilon} f(x)dx.$$

(b) Let f be integrable in $(a,b]$ and $\lim_{x\to a+}|f(x)| = \infty$. The improper integral is defined as

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0+}\int_{a+\epsilon}^{b} f(x)dx.$$

(c) Let f be integrable in (a,b), $\lim_{x\to a+}|f(x)| = \infty$ and $\lim_{x\to b-}|f(x)| = \infty$. The improper integral is defined as

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0+}\int_{a+\epsilon}^{c} f(x)dx + \lim_{\epsilon\to 0+}\int_{c}^{b-\epsilon} f(x)dx,$$

where $a < c < b$.

(d) Let f be integrable in $[a,c)$ and $(c,b]$ and $\lim_{x\to c\pm}|f(x)| = \infty$. The improper integral is defined as

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0+}\int_{a}^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0+}\int_{c+\epsilon}^{b} f(x)dx.$$

We have defined the improper integral as the limit of a proper definite integral. There are three different outcomes of the limit process:

(i) If the limit does not exist but is also not infinite (for (c) and (d), in at least one limit), the improper integral does not exist.
(ii) If the limit does not exist and is $+\infty$ or $-\infty$ (for (c) and (d), in at least one limit and if the other limit is not covered by (i)), the improper integral is divergent with $I = +\infty$ or $I = -\infty$.
(iii) If the limit exists and is finite (for (c) and (d), in both limits), the improper integral is convergent and its value is I.

Example 6:
Type (a)

$$I = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{\epsilon\to 0+} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx = \lim_{\epsilon\to 0+} [\arcsin x]_0^{1-\epsilon},$$

and therefore

$$I = \lim_{\epsilon\to 0+} (\arcsin(1-\epsilon) - \arcsin 0) = \arcsin 1 = \frac{\pi}{2} \qquad \text{convergent.}$$

We also note that the function is not defined at $x = 1$, and, for example, the integral $\int_3^7 \frac{1}{\sqrt{1-x^2}}$ would be a standard definite integral.

Type (c)

$$I = \int_{-1}^1 \frac{1}{\sqrt{1-x^2}}dx = \lim_{\epsilon\to 0+} \int_{-1+\epsilon}^{c} \frac{1}{\sqrt{1-x^2}}dx + \lim_{\epsilon\to 0+} \int_{c}^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx.$$

We have to fix c and appropriately choose $c = 0$ because it is in the center of the interval. So we have

$$I = \lim_{\epsilon\to 0+} \int_{-1+\epsilon}^{0} \frac{1}{\sqrt{1-x^2}}dx + \lim_{\epsilon\to 0+} \int_{0}^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx.$$

The second term we have computed above, as well as the antiderivative and obtain

$$I = \frac{\pi}{2} + \lim_{\epsilon\to 0+} [\arcsin x]_{-1+\epsilon}^0 = \frac{\pi}{2} + \lim_{\epsilon\to 0+} (0 - \arcsin(\epsilon - 1)) = \frac{\pi}{2} - \arcsin(-1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

In this example, we could have used also the even symmetry of the function to obtain the result. A separate example for type (b) is not necessary.

Type (d)

$$I = \int_a^b \frac{1}{x-c}dx \qquad \text{with} \qquad a < c < b.$$

The integrand is not defined for $x = c$ and $\lim_{x\to c\pm} \frac{1}{x-c} = \pm\infty$ and we write

$$I = \int_a^b f(x)dx = \lim_{\epsilon\to 0+} \int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0+} \int_{c+\epsilon}^{b} f(x)dx.$$

We have to calculate both terms separately

$$\begin{aligned}
\lim_{\epsilon\to 0+}\int_a^{c-\epsilon} f(x)dx &= \lim_{\epsilon\to 0+}\left[\ln|x-c|\right]_a^{c-\epsilon} = \lim_{\epsilon\to 0+}\left(\ln|c-\epsilon-c| - \ln|a-c|\right) \\
&= \lim_{\epsilon\to 0+}\ln\frac{\epsilon}{|a-c|} = -\infty \quad \text{divergent.} \\
\lim_{\epsilon\to 0+}\int_{c+\epsilon}^{b} f(x)dx &= \lim_{\epsilon\to 0+}\left[\ln|x-c|\right]_{c+\epsilon}^{b} = \lim_{\epsilon\to 0+}\left(\ln|b-c| - \ln|c+\epsilon-c|\right) \\
&= \lim_{\epsilon\to 0+}\ln\frac{|b-c|}{\epsilon} = \infty \quad \text{divergent.}
\end{aligned}$$

While both improper integrals diverge, and so the overall integral, we may recognize that the curve of the function has an odd symmetry with respect to $x = c$ and that the divergent integrals may cancel out if the limit process is combined. This is indeed the case for this example:

$$\begin{aligned}
\text{P.V.} \int_a^b \frac{1}{x-c}dx &= \lim_{\epsilon\to 0+}\left[\int_a^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{b} f(x)dx\right] \\
&= \ln\frac{|c-\epsilon-c|}{|a-c|} + \ln\frac{|b-c|}{|c+\epsilon-c|} = \ln\frac{|b-c|}{|a-c|},
\end{aligned}$$

This is called the *principal value* of the integral.

It is possible to combine improper integrals of the first and second kind. In that case, limits have to be taken separately.

9.6 Applications of integrals

The application of integral calculus are very diverse and are present in many fields of sciences and engineering. We limit ourselves here to two general and simple concepts: to determine areas and lengths of curves.

9.6.1 The area between two curves

The Riemann integral introduces the definite integral precisely as a representation of the area A between the curve of a bound function and the abscissa in an interval $[a, b]$. For the integral really representing the area, we assume that $f(x) > 0$ for all $x \in [a, b]$. In that case, $\int_a^b f(x)\,dx > 0$ and it corresponds directly to the area (if the calculation contains units of length, the integral contains units of lengths squared).

We have to check what happens if $f(x)$ is negative or if it changes its sign in the interval $[a,b]$. Let $I = \int_a^b f(x)\,dx$

$$f(x) \geq 0 \quad \forall \quad x \in [a,b] \qquad A = I = \int_a^b f(x)\,dx,$$
$$f(x) \leq 0 \quad \forall \quad x \in [a,b] \qquad A = -I = \int_a^b (-f(x))\,dx,$$

i.e., the integral of a negative function is negative, but its absolute value corresponds to the area (recall that que $\forall$ means "for all the values of").

If $f(x)$ changes its sign in points $c_1, c_2, \ldots, c_n$ (that can be identified solving $f(c_i) = 0$ for c_i), then it is possible to determine the areas of each subinterval $[a, c_1], [c_1, c_2], \ldots, [c_n, b]$ according to the rules above and sum them up, or use the absolute value:

$$A = \int_a^b |f(x)|\,dx.$$

We can generalize the concept to the area between two curves given by $f(x)$ and $g(x)$:

$$f(x) - g(x) \geq 0 \quad \forall \quad x \in [a,b] \qquad A = I = \int_a^b (f(x) - g(x))\,dx,$$
$$f(x) - g(x) \leq 0 \quad \forall \quad x \in [a,b] \qquad A = -I = \int_a^b (g(x) - f(x))\,dx,$$

If $f(x) - g(x)$ changes its sign in points $c_1, c_2, \ldots, c_n$ (that can be found solving $f(c_i) = g(c_i)$), then it is possible to determine the areas of each subinterval $[a, c_1], [c_1, c_2], \ldots, [c_n, b]$ according to the rules above and sum them up, or use the absolute value:

$$A = \int_a^b |f(x) - g(x)|\,dx.$$

9.6.2 The arclength

A curve in the cartesian plane can be represented in various ways. We consider two types, the explicit form and the parametric form.

If a curve is given by a function $f(x)$ explicitly, we identify its position by sets of pairs $(x, y) = (x, f(x))$ and assume that $f(x)$ is differentiable and its derivative continuous in the compact interval $[a, b]$. Then, the arclength of the curve between two points $(a, f(a))$ and $(b, f(b))$ is given by

$$l = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

If a curve is given in parametric form, we identify its position by sets of pairs $(x, y) = (x(t), y(t))$, where both the position on the abscissa and the ordinate depend on the parameter t. We assume that both $x(t)$ and $y(t)$ are differentiable and their derivatives continuous in the compact interval $[t_1, t_2]$. Then, the arclength of the curve between two points $(x(t_1), y(t_1))$ and $(x(t_2), y(t_2))$ is given by

$$l = \int_{t_1}^{t_2} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

Example 7: Calculate the arclength of the following curves:

(a) Let $f(x) = \frac{4}{3}x$ between $x = 0$ and $x = 3$.
The curve is a straight line with slope $4/3$ passing through the origin, limited by $(0,0)$ and $(3,4)$. Easily, we obtain $f'(x) = \frac{4}{3}$. We use the formula for the explicit function and determine

$$l = \int_0^3 \sqrt{1 + (f'(x))^2}\, dx = \int_0^3 \sqrt{1 + \frac{16}{9}}\, dx = \int_0^3 \sqrt{\frac{25}{9}}\, dx = \int_0^3 \frac{5}{3}\, dx = \frac{5}{3}[x]_0^3 = 5.$$

(b) Let $x(t) = rt - r\sin t$ and $y(t) = r - r\cos t$ between $t = 0$ and $t = 2\pi$.
The curve represents the arc of a cycloid. In $t = 0$, $(x, y) = (0, 0)$ and in $t = 2\pi$, $(x, y) = (2\pi r, 0)$.
We calculate:

$$\begin{aligned} x'(t) &= r(1 - \cos t), \\ y'(t) &= r\sin t, \end{aligned}$$

and therefore

$$\begin{aligned} l &= \int_0^{2\pi} \sqrt{(r(1-\cos t))^2 + (r\sin t)^2}\, dt = r\int_0^{2\pi} \sqrt{(1-\cos t)^2 + \sin^2 t}\, dt \\ &= r\int_0^{2\pi} \sqrt{1 + \cos^2 t - 2\cos t + \sin^2 t}\, dt = r\int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt. \end{aligned}$$

Now, we use the trigonometric identity $2\sin^2 u = 1 - \cos(2u)$:

$$l = r\int_0^{2\pi} \sqrt{4\sin^2\left(\frac{t}{2}\right)}\, dt = 2r\int_0^{2\pi} \sin\left(\frac{t}{2}\right)\, dt = 2r\int_0^{\pi} \sin p \cdot 2\, dp,$$

where we do not need to take the absolute value of $\sin(\frac{t}{2})$ because in the relevant interval the sine is not negative and we do a change of variable $t/2 = p$. We close the calculation:

$$l = 4r\int_0^{\pi} \sin p\, dp = 4r[-\cos p]_0^{\pi} = -4r[\cos p]_0^{\pi} = -4r(-1-1) = 8r.$$