Chapter 8: Taylor Polynomial

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8 Taylor polynomial

In this chapter we study a very useful method to approximate functions, obtain limits, among other uses: the Taylor polynomial.

8.1 Polynomial approximation of functions

Theorem 1 (Weierstrass' approximation theorem):
Let f be a continuous function on $[a,b]$. Then, $\forall \epsilon > 0, \exists$ a polynomial $P(x)$ such that

$$|f(x) - P(x)| < \epsilon \quad \forall x \in [a,b].$$

This is a remarkable statement: it means that an *arbitrary* continuous function can be approximated *to arbitrary precision* by a polynomial! This is visualized in Fig. 1.

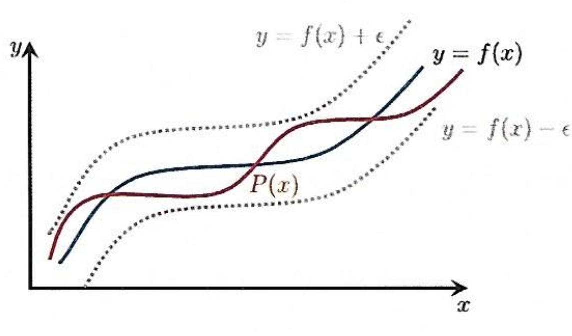

Figure 1: Illustration of Weierstrass' approximation theorem: For any $\epsilon > 0$, there is a polynomial $P(x)$ such that it uniformly close to $f(x)$ in an interval $[a,b]$.

By approximating an arbitrary function by a polynomial, it is useful to state its analytical properties. For that the following definition comes in handy:

Definition 1 (differentiability classes):
A function f is of differentiability class C^k if the derivatives f', f'', ..., $f^{(k)}$ exist and are continuous.

Comments:
(1) Since differentiability includes continuity, continuity is implied for all derivatives except for $f^{(k)}$.
(2) If a function has derivatives of any order, it is called infinitely (often) differentiable, smooth, or of class C^∞.
(3) Depending on whether the function is defined on an open or closed interval, there are slightly different meanings, in particular implying lateral differentiability on the endpoints of a closed interval.

Theorem 2 (polynomials are smooth):
Let P be a polynomial of order n. Then, P is infinitely (often) differentiable in $\mathbb{R}$.

Since differentiability includes continuity, it is not necessary to state that P is continuous (although it is true). We already knew that a polynomial is differentiable. What is new here is the generalization to higher derivatives. As we know, the derivative of a polynomial of order n yields a polynomial of order $n-1$. Therefore, differentiating a polynomial (of order n) n times produces a constant. Then, any further derivatives (and there is nothing preventing us from taking higher derivatives) become zero.

8.2 Taylor's theorem

The following theorem provides a very useful tool to represent sufficiently often differentiable functions by polynomials, as it not only provides the approximation, but also a way to evaluate the goodness of the approximation.

Theorem 3 (Taylor's theorem):
Let f be of class C^n on the interval $[a,b]$ and suppose that the $(n+1)$-th derivative of f exists on (a,b). Let $x_0 \in [a,b]$. Then, $\forall x \in (a,b)$, $\exists c$ between x and x_0 such that

$$f(x) = P_{n,x_0}(x) + R_{n,x_0}(x),$$

with $P_{n,x_0}(x)$ being the *Taylor polynomial* and $R_{n,x_0}(x)$ the *remainder*, given by

$$\begin{aligned} P_{n,x_0}(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n, \\ R_{n,x_0}(x) &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned}$$

Comments:
(1) This is also called the Taylor expansion of f about (around) x_0.
(2) The specific value x_0 is the point about which we approximate the function and is

typically given. If $x_0 = 0$, the polynomial is also called the Maclaurin polynomial.
(3) The dependence on x resides solely in the polynomial terms $(x - x_0)^k$ since $f(x_0)$, $f'(x_0)$, $f''(x_0)$,... are all scalar values. To approximate (replace) a function f by its Taylor polynomial it is necessary to compute the aforementioned derivatives.
(4) The quality of the approximation (or the goodness of the fit) can be assessed by calculate the remainder term (if possible) or to give an upper bound to it.
(5) The quality of the approximation typically increases if (a) the order of the Taylor polynomial increases and (b) if x approaches x_0. Obviously, for $x = x_0$, the Taylor polynomial is simply $f(x_0)$, showing that Taylor's theorema is only useful for $x \neq x_0$, and hence an appropriate choice of x_0 is important (not too far from the values of x where f should be evaluated).
(6) This formulation of $R_{n,x_0}(x)$ is called the Lagrange remainder. There are other variants, namely the Cauchy form and the integral form (not shown here). It is important to note that in general there is no way to know what specific value c takes.
(7) There are nice animations of Taylor polynomial approximations at:

`https://en.wikipedia.org/wiki/Taylor's_theorem`

8.3 Applications of Taylor's theorem

The fundamental application of Taylor's theorem is to approximate (replace) a possibly complicated function by a polynomial. Polynomials are C^∞ functions and very "well-behaved", as it is easy to compute any derivative of it, obtain its graph, etc. The remainder term can be used to check the quality of the approximation. Among other uses, Taylor's theorem can be used to evaluate limits and proving inequalities.

Example 1:
(a) Approximate $f(x) = \sin(x)$ by its Taylor polynomial of order 3 about $x_0 = 0$. (b) Then give an upper bound to the absolute error in the interval $[0, 1]$. (c) Compare with the actual absolute error at $x = 0.5$.

(a) We apply Taylor's theorem as indicated:

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(c)}{4!}(x - x_0)^4 \\ &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f^{(3)}(0)}{6}x^3 + \frac{f^{(4)}(c)}{24}x^4. \end{aligned}$$

We have to compute derivatives up to 4th order:

	x_0 or c	$x_0 = 0$
f	$\sin(x_0)$	0
f'	$\cos(x_0)$	1
f''	$-\sin(x_0)$	0
$f^{(3)}$	$-\cos(x_0)$	-1
$f^{(4)}$	$\sin(c)$	

With this we obtain for the Taylor polynomial:

$$P_{3,0}(x) = x - \frac{1}{6}x^3,$$

and, overall:

$$f(x) \quad = \quad x - \frac{1}{6}x^3 + \frac{\sin(c)}{24}x^4,$$

where c is between $x_0 = 0$ and x.

(b) In principle, x can be any number, but we know that $\sin(x)$ is bound between -1 and 1 while the polynomial will go to ∞ for $x \to \infty$, and the approximation will break down. This example limits x to the interval $[0, 1]$ and we use the remainder term to determine the upper bound to the absolute error in that interval:

$$max(|R_{3,0}(x)|) = max\left(\left|\frac{\sin(c)}{24}x^4\right|\right),$$

where the maximum is taken over all values $x \in [0, 1]$. The absolute value is taken because we are interested in the absolute error of the approximation (and do not care whether the difference $f(x) - P_{n,x_0}(x)$ is positive or negative).

In the interval $[0, 1]$, x^4 takes its maximum at $x = 1$. The same holds for the sine function (c is between 0 and 1), and therefore

$$max(|R_{3,0}(x)|) = \left|\frac{\sin(1)}{24}\right| \approx 0.0351 \quad \text{(4 d.p.)}.$$

Result: Between $[0, 1]$, $\sin(x)$ can be approximated by $x - \frac{1}{6}x^3$ with a maximum absolute error of 0.0351 (to a precision of 4 decimal places).

A comment may be due here: It would be possible to obtain a weaker (larger) upper bound by using that $|\sin x| \leq 1$ for all $x \in \mathbb{R}$:

$$max(|R_{3,0}(x)|) = \left|\frac{1}{24}\right| \approx 0.0417 \quad \text{(4 d.p.)}.$$

For example, if the interval given was $[0, b]$, with $b \in \mathbb{R}$, we would have to use $|\sin x| \leq 1$ and keep the term b^4:

$$max(|R_{3,0}(x)|) = \left|\frac{1}{24}b^4\right|.$$

This result tells us that the approximation $f(x) \approx P_{3,0}(x)$ becomes bad if $b \gg 1$.

(c) We have to compare $f(0.5)$ and $P_{3,0}(0.5)$:

$$\begin{aligned} f(0.5) &= \sin(0.5) = 0.479426 \quad (6 \text{ d.p.}) \\ P_3(0.5) &= 0.5 - \frac{0.5^3}{6} = \frac{23}{48} = 0.479167 \quad (6 \text{ d.p.}) \end{aligned}$$

The actual absolute error is

$$E(0.5) = |0.479426 - 0.479167| = 0.000259 = 2.59 \times 10^{-4}.$$

This number is smaller than $max(|R_{3,0}(x)|)$, as expected.

Example 2:
Calculate the Taylor polynomial of order 3 of $f(x) = x^3$ about $x_0 = 2$ together with the remainder term.

We write

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \frac{f^{(4)}(c)}{4!}(x - x_0)^4 \\ &= f(2) + f'(2)(x - 2) + \frac{f''(2)}{2}(x - 2)^2 + \frac{f^{(3)}(2)}{6}(x - 2)^3 + \frac{f^{(4)}(c)}{24}(x - 2)^4. \end{aligned}$$

We have to compute derivatives up to 4th order:

	x_0 or c	$x_0 = 2$
f	x_0^3	8
f'	$3x_0^2$	12
f''	$6x_0$	12
$f^{(3)}$	6	6
$f^{(4)}$	0	

With this we obtain:

$$\begin{aligned} P_{3,2}(x) &= 8 + 12(x - 2) + \frac{12}{2}(x - 2)^2 + \frac{6}{6}(x - 2)^3 \\ &= 8 + 12(x - 2) + 6(x - 2)^2 + (x - 2)^3, \\ R_{3,2}(x) &= \frac{0}{24}(x - 2)^4 = 0. \end{aligned}$$

The remainder term is zero, and consequently we have $f(x) = x^3 = P_{3,2}(x)$. This can be easily checked by expanding the terms of $P_{3,2}$. Furthermore, if one chooses $x_0 = 0$ in this example, one directly calculates $P_{3,0}(x) = x^3$. This reflects a general result: the Taylor polynomial of order n of a polynomial of order n are identical.

Example 3:
Calculate the following limit using the Taylor expansions of the the involved functions:

$$\lim_{x \to 0} \frac{\cos x - e^x + x}{x^2}.$$

Substitution of $x=0$ yields an indetermination $0/0$. Since the denominator is x^2, we develop the functions $\cos x$ and e^x up to that order, using $x_0=0$ since this is the point at which the limit is taken.

$$\begin{aligned}
\cos x &= \cos(0)+(\cos x)'(0)\cdot(x-0)+\frac{(\cos x)''(0)}{2}(x-0)^2+h.o.t.\\
&= 1-\frac{1}{2}x^2+h.o.t.,\\
e^x &= e^0+(e^x)'(0)\cdot(x-0)+\frac{(e^x)''(0)}{2}(x-0)^2+h.o.t.\\
&= 1+x+\frac{1}{2}x^2+h.o.t.,
\end{aligned}$$

where $h.o.t.$ stands for *higher order terms*. Now, we proceed

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}=\lim_{x\to 0}\frac{1-\frac{1}{2}x^2-1-x-\frac{1}{2}x^2+x+h.o.t.}{x^2}=\lim_{x\to 0}\frac{-x^2+h.o.t.}{x^2}$$

This simplifies to

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}=-1+\lim_{x\to 0}\frac{h.o.t.}{x^2},$$

but as we know the higher order terms contain contributions proportional to $x^3+\cdots$, implying that the latter limit is zero and we have as result

$$\lim_{x\to 0}\frac{\cos x-e^x+x}{x^2}=-1.$$

Example 4:
We approximate $f(x)=e^x$ in the interval $[0,1]$ by a Taylor polynomial. Of which order needs the polynomial to be for the absolute error to be smaller than 0.05?

We select $x_0=0$ and the Taylor polynomial of e^x becomes

$$P_{n,0}(x) \;=\; 1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\cdots+\frac{1}{n!}x^n,$$

but what we really need to respond the question is the remainder term

$$R_{n,0}(x)=\frac{e^c}{(n+1)!}x^{n+1},$$

where c is between 0 and x and we have to solve for n

$$max(|R_{n,0}(x)|)=max\left(\left|\frac{e^c}{(n+1)!}x^{n+1}\right|\right)<0.05.$$

Since x is limited to 1 and the exponential function is strictly increasing, the maximum value of e^c in the interval $[0,1]$ is e^1. The power function x^{n+1} also is strictly increasing in the interval $[0, 1]$ and takes its maximum value at $x = 1$. We obtain

$$\begin{aligned} \frac{e}{(n+1)!} 1^{n+1} &< 0.05, \\ \frac{1}{(n+1)!} &< \frac{0.05}{e}, \\ (n+1)! &> \frac{e}{0.05} \approx 54.4. \end{aligned}$$

Since $4! = 24$ and $5! = 120$, we conclude that we have to approximate e^x at least by $P_{4,0}(x)$ (for the absolute error in $[0, 1]$ to be smaller than 0.05).

More examples see the exercise sheet.