

Problems

Problem 7.1 Let f and g be differentiable functions in $\mathbb{R}$. Write down the derivative of the following functions in their respective domains:

- (i) $h(x) = \sqrt{f(x)^2 + g(x)^2}$;
- (ii) $h(x) = \arctan\left(\frac{f(x)}{g(x)}\right)$;
- (iii) $h(x) = f(g(x))e^{f(x)}$;
- (iv) $h(x) = \log(g(x) \sin f(x))$;
- (v) $h(x) = f(x)^{g(x)}$;
- (vi) $h(x) = \frac{1}{\log(f(x) + g(x)^2)}$.

Problem 7.2

- (a) Make up a continuous function in $\mathbb{R}$ which vanishes for $|x| \geq 2$ and equals 1 for $|x| \leq 1$.
- (b) Do it again, but this time make sure that the function is differentiable in $\mathbb{R}$.

Problem 7.3 Check that the following functions satisfy the specified differential equations, where c , c_1 , and c_2 are constants:

- (i) $f(x) = \frac{c}{x}$ satisfies $xf' + f = 0$;
- (ii) $f(x) = x \tan x$ satisfies $xf' - f - f^2 = x^2$;
- (iii) $f(x) = c_1 \sin 3x + c_2 \cos 3x$ satisfies $f'' + 9f = 0$;
- (iv) $f(x) = c_1 e^{3x} + c_2 e^{-3x}$ satisfies $f'' - 9f = 0$;
- (v) $f(x) = c_1 e^{2x} + c_2 e^{5x}$ satisfies $f'' - 7f' + 10f = 0$;
- (vi) $f(x) = \log(c_1 e^x + e^{-x}) + c_2$ satisfies $f'' + (f')^2 = 1$.

Problem 7.4 Prove the identities (valid only in the specified regions)

- (i) $\arctan x + \arctan \frac{1}{x} = \frac{\pi}{2}$, for $x > 0$;
- (ii) $\arctan \frac{1+x}{1-x} - \arctan x = \frac{\pi}{4}$, for $x < 1$;
- (iii) $2 \arctan x + \arcsin \frac{2x}{1+x^2} = \pi$, for $x \geq 1$.

HINT: Differentiate the equation and check one point of the specified region.

Problem 7.5 At which points does the graph of the function $f(x) = x + (\sin x)^{1/3}$ has a vertical tangent?

Problem 7.6 Given the function

$$f(x) = \begin{cases} \frac{x}{1 + e^{1/x}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

calculate the angle between the tangents on the left and on the right of its graph at $x = 0$.

Problem 7.7 Find the sets where the function $f(x) = \sqrt{x+2} \arccos(x+2)$ is continuous and differentiable.

Problem 7.8 Calculate the smallest α for which $f(x) = |\alpha x^2 - x + 3|$ is differentiable in $\mathbb{R}$.

Problem 7.9 Prove that the function

$$f(x) = \begin{cases} \frac{1}{x} - \frac{1}{e^x - 1}, & \text{if } x \neq 0, \\ \frac{1}{2}, & \text{if } x = 0, \end{cases}$$

is differentiable at $x = 0$ calculating $f'(0)$.

Problem 7.10 Given the function

$$f(x) = \begin{cases} a + bx^2, & |x| \leq c, \\ \frac{1}{|x|}, & |x| > c, \end{cases} \quad c > 0,$$

find a and b so that it is continuous and differentiable in $\mathbb{R}$.

Problem 7.11 Given the function

$$f(x) = \begin{cases} \frac{3-x^2}{2}, & x < 1, \\ \frac{1}{x}, & x \geq 1, \end{cases}$$

- (a) determine the sets where it is continuous and where it is differentiable;
- (b) check that the mean value theorem can be applied to this function in $[0, 2]$ by determining the point(s) $c \in (0, 2)$ where the theorem holds.

Problem 7.12 Function $f(x) = 1 - x^{2/3}$ vanishes in $x = \pm 1$; however $f'(x) \neq 0$ in $(-1, 1)$. Find which hypothesis of Rolle's theorem is not satisfied.

Problem 7.13 Prove, using Rolle's theorem, the following statements about a function f that is continuous in $[a, b]$ and differentiable in (a, b) :

- (i) If f vanishes $k(\geq 2)$ times in $[a, b]$ then f' vanishes at least $k - 1$ times in $[a, b]$.
- (ii) If f is n -times differentiable in (a, b) and vanishes in $n + 1$ different points of $[a, b]$, then $f^{(n)}$ vanishes at least once in $[a, b]$.

Problem 7.14 Using the mean value theorem, find an approximation to $26^{2/3}$ and $\log(3/2)$.

Problem 7.15 Calculate the limits

- (i) $\lim_{x \rightarrow 0} \frac{e^x - \sin x - 1}{x^2};$
- (ii) $\lim_{x \rightarrow 0} \frac{\log |\sin 7x|}{\log |\sin x|};$
- (iii) $\lim_{x \rightarrow 1^+} \log x \log(x - 1);$
- (iv) $\lim_{x \rightarrow \infty} x^{1/x};$
- (v) $\lim_{x \rightarrow 0} \frac{(1+x)^{1+x} - 1 - x - x^2}{x^3};$
- (vi) $\lim_{x \rightarrow \infty} x \left(\tan \frac{2}{x} - \tan \frac{1}{x} \right).$

Problem 7.16 Suppose $h(x)$ is a twice-differentiable function and let

$$f(x) = \begin{cases} \frac{h(x)}{x^2}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Calculate $h(0)$, $h'(0)$, and $h''(0)$ so that f is continuous.

Problem 7.17 Calculate the limits

- (i) $\lim_{x \rightarrow \infty} x \left[\left(1 + \frac{1}{x} \right)^x - e \right];$
- (ii) $\lim_{x \rightarrow \infty} \frac{(1 + \frac{1}{x})^{x^2}}{e^x};$
- (iii) $\lim_{x \rightarrow \infty} \left(\frac{2^{1/x} + 18^{1/x}}{2} \right)^x;$
- (iv) $\lim_{x \rightarrow \infty} \left(\frac{1}{p} \sum_{k=1}^p a_k^{1/x} \right)^x, \text{ with } p \in \mathbb{N} \text{ and } a_k > 0.$

Problem 7.18 If f is a differentiable function such that

$$\lim_{x \rightarrow 0} \frac{f(2x^3)}{5x^3} = 1$$

and its derivative f' is continuous at $x = 0$,

- (a) calculate $f(0)$;
- (b) calculate $f'(0)$;
- (c) calculate $\lim_{x \rightarrow 0} \frac{(f \circ f)(2x)}{f^{-1}(3x)}$.

Problem 7.19 The equation $e^{-f} f' = 2 + \tan x$ together with the condition $f(0) = 1$ define a one-to-one, differentiable function in the interval $[-\pi/4, \pi/4]$. If $g(x) = f^{-1}(x+1)$, calculate the limit

$$\lim_{x \rightarrow 0} \frac{e^x - e^{-\sin x}}{g(x)}.$$

Problem 7.20 Let $f(x) = |x^3(x-4)| - 1$.

- (a) Find where f is continuous and where it is differentiable.
- (b) Determine its extrema.
- (c) Prove that $f(x) = 0$ has a unique solution in $[0, 1]$.

Problem 7.21 Solve these optimisation problems:

- (a) A factory that produces tomato sauce wants to can it in cylindrical cans of a fixed volume V . Determine their radius r and height h so that their fabrication consumes the least possible material.
- (b) A recipient with square bottom and no cap must be covered by a thin layer of lead. If the volume of the recipient must be 32 litres, which dimensions should it have so that it requires the least possible amount of lead?
- (c) Find two numbers $x, y > 0$ such that $x + y = 20$ and $x^2 y^3$ is maximum.
- (d) Find the rectangle inscribed in the ellipse $(x/a)^2 + (y/b)^2 = 1$ with its sides parallel to the axes of the ellipse, such that its area is maximum.
- (e) With a tangent to the parabola $y = 6 - x^2$ and the positive axes one can make a triangle. Determine which of those triangles has the smallest area and compute it.
- (f) We need to construct a box with no cap with the shape of a parallelepiped whose base is an equilateral triangle, and whose volume is 128 cm^3 . If the material for the base costs 0.20 euros/cm² and that for the lateral surfaces costs 0.10 euros/cm², what are the dimensions of the cheapest such box?
- (g) A right triangle ABC has vertex A at the origin, vertex B on the circumference $(x-1)^2 + y^2 = 1$ —side AB is the hypotenuse of the triangle—and side AC on the horizontal axis. Calculate the location of C that maximises the area of the triangle.
- (h) Let $P = (x_0, y_0)$ be a point of the first quadrant ($x_0, y_0 > 0$). A straight line through P cuts the axes at $A = (x_0 + \alpha, 0)$ and $B = (0, y_0 + \beta)$. Calculate $\alpha > 0$ and $\beta > 0$ so as to minimise
 - (i) the length of segment AB;
 - (ii) the sum of the lengths of OA and OB;
 - (iii) the area of the triangle OAB.

HINT: Triangle similarity implies $\beta = x_0 y_0 / \alpha$.

Problem 7.22 Prove the following inequalities:

- (a) $(1+x)^a \geq 1+ax$ for all $a \geq 1, x > -1$ (Bernoulli's inequality);
- (b) $e^x \geq 1+x$ for all $x \in \mathbb{R}$;
- (c) $\frac{x}{1+x} \leq \log(1+x) \leq x$ for all $x > -1$.

HINT: In all cases try to minimise the appropriate function.

Problem 7.23

- (a) Prove that $\frac{\log x}{x} < \frac{1}{e}$ for all $x > 0, x \neq e$.

(b) Prove that the previous inequality is equivalent to $e^x > x^e$ for all $x > 0$, $x \neq e$.

Problem 7.24 Determine the number of solutions of the following equations in the specified domains:

(i) $x^7 + 4x = 3$ in $\mathbb{R}$;

(iii) $x^4 - 4x^3 = 1$ in $\mathbb{R}$;

(v) $x^x = 2$ in $[1, \infty)$;

(ii) $x^5 = 5x - 6$ in $\mathbb{R}$;

(iv) $\sin x = 2x - 1$ in $\mathbb{R}$;

(vi) $x^2 = \log \frac{1}{x}$ in $(1, \infty)$.