Métodos Matemáticos de Bioingeniería

Grado en Ingeniería Biomédica

Lecture 2

Marius A. Marinescu

Departamento de Teoría de la Señal y Comunicaciones
Área de Estadística e Investigación Operativa
Universidad Rey Juan Carlos

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Outline

Outline

Dot and Cross Product

- When we introduced the arithmetic operations,

  Why the product of two vectors
  was not defined?

- **Vector multiplication** could be defined in a manner analogous to the vector addition:

  By componentwise multiplication.

- However, such a definition is not very useful in our context.
- Instead, we shall define and use two different concepts of a product of two vectors:
  - The Euclidean **inner product**, or **dot product**, defined for two vectors in $\mathbb{R}^n$ (where n is arbitrary).
  - The **cross** or **vector product**, defined only for vectors in $\mathbb{R}^3$.

Definition 3.1

- Let $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors.
- The **dot** (or **inner** or **scalar**) **product** of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Dot product takes two vectors
and produces a single real number
(not a vector)

Example 1

In $\mathbb{R}^3$ we have

$$\begin{aligned} (1, -2, 5) \cdot (2, 1, 3) &= (1)(2) + (-2)(1) + (5)(3) = 15 \\ (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (\mathbf{i} - 2\mathbf{k}) &= (3)(1) + (2)(0) + (-1)(-2) = 5 \end{aligned}$$

Properties of Dot Products

If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are any vectors in $\mathbb{R}^n$, and $k \in \mathbb{R}$ is any scalar:

1. $\mathbf{a} \cdot \mathbf{a} \geq 0$, and $\mathbf{a} \cdot \mathbf{a} = 0$ if and only if $\mathbf{a} = \mathbf{0}$.
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (**commutative** property)
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (**distributive** property)
4. $(k\mathbf{a}) \cdot \mathbf{b} = k(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (k\mathbf{b})$

Definition 3.2

- If $\mathbf{a} = (a_1, a_2, a_3)$ then the **length** of $\mathbf{a}$ (also called the **norm** or **magnitude**) is

$$\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

- Using the distance formula, the length of the arrow from the origin to (a_1, a_2, a_3) is

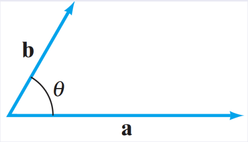
$$dist(\mathbf{a}, \mathbf{0}) = \sqrt{(a_1 - 0)^2 + (a_2 - 0)^2 + (a_3 - 0)^2}$$

- Thus,

$$\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2 \text{ or } \|\mathbf{a}\| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$$

Theorem 3.3

Let $\mathbf{a}$ *and* $\mathbf{b}$ *be two nonzero vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*) drawn with their tails at the same point and let* θ*, where* $0 \leq \theta \leq \pi$*, be the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *,*



Then,

$$\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$$

Note

- If either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, then θ is indeterminate (i.e., can be any angle).

Demonstration on blackboard. □

Corollary of Theorem 3.3

- Theorem 3.3 may be used to find the **angle between two nonzero vectors** $\mathbf{a}$ and $\mathbf{b}$

$$\theta = \cos^{-1} \frac{\mathbf{a} \cdot \mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$$

- The use of the inverse cosine is unambiguous, since we take $0 \leq \theta \leq \pi$

Example 2

- If $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} - \mathbf{k}$, then formula gives

$$\theta = \cos^{-1} \frac{(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{j} - \mathbf{k})}{\|\mathbf{i} + \mathbf{j}\| \|\mathbf{j} - \mathbf{k}\|} = \cos^{-1} \frac{1}{(\sqrt{2} \cdot \sqrt{2})} = \cos^{-1} \frac{1}{2} = \frac{\pi}{3}$$

Orthogonality

- If $\mathbf{a}$ and $\mathbf{b}$ are nonzero, v then **Theorem 3.3** implies

$$\cos\theta = 0 \text{ if and only if } \mathbf{a}\cdot\mathbf{b} = 0$$

- We have $\cos\theta = 0$ just in case $\theta = \frac{\pi}{2}$

Remember that $0 \leq \theta \leq \pi$

- We call $\mathbf{a}$ and $\mathbf{b}$ **perpendicular** (or **orthogonal**) when $\mathbf{a}\cdot\mathbf{b} = 0$
- If either $\mathbf{a}$ or $\mathbf{b}$ is the zero vector, the angle θ is undefined
- Since $\mathbf{a}\cdot\mathbf{b} = 0$ if $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we adopt the standard convention

The zero vector
is perpendicular to every vector

Example 3

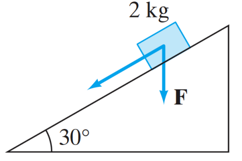
- The vector $\mathbf{a} = \mathbf{i} + \mathbf{j}$ is orthogonal to the vector $\mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$

$$(\mathbf{i} + \mathbf{j}) \cdot (\mathbf{i} - \mathbf{j} + \mathbf{k}) = (1)(1) + (1)(-1) + (0)(1) = 0$$

Outline

Motivation example

- Suppose that a 2 kg object is sliding down a ramp.
- The ramp has a 30° inclination with the horizontal:



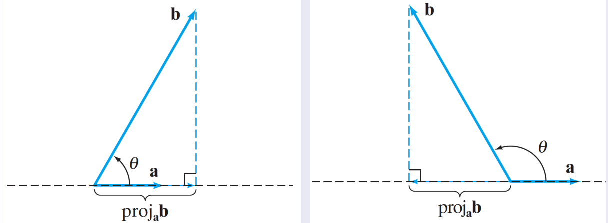
- If we neglect friction, the only force acting on the object is gravity.

What is the component of the gravitational force in the direction of motion of the object?

- To answer questions of this nature, we need to find the **projection** of one vector on another.

Projection of one vector on another: intuitive idea

- Let $\mathbf{a}$ and $\mathbf{b}$ be two nonzero vectors. v
- Imagine dropping a perpendicular line from the head of $\mathbf{b}$ to the line through $\mathbf{a}$.



- The **projection of $\mathbf{b}$ onto $\mathbf{a}$** , denoted proj$_\mathbf{a}\mathbf{b}$, is the vector represented by the tiny arrow in figure.

Projection of one vector on another: precise formula

- Recall that

A vector is determined by
magnitude (length) and direction

- The direction of $\text{proj}_{\mathbf{a}}\mathbf{b}$ is either
  - The same as that of $\mathbf{a}$ or
  - Opposite to $\mathbf{a}$ if the angle θ between $\mathbf{a}$ and $\mathbf{b}$ is more than $\frac{\pi}{2}$
- Using trigonometry

$$|\cos\theta| = \frac{\|\text{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|}$$

- The absolute value sign around $\cos\theta$ is needed in case

$$\frac{\pi}{2} \leq \theta \leq \pi$$

Projection of one vector on another: precise formula

- Since,

$$|\cos\theta| = \frac{\|\mathsf{proj}_{\mathbf{a}}\mathbf{b}\|}{\|\mathbf{b}\|}$$

- with a bit of algebra and using that $|\mathbf{a}\cdot\mathbf{b}| = \|\mathbf{a}\|\|\mathbf{b}\||\cos\theta|$, we have

$$\|\mathsf{proj}_{\mathbf{a}}\mathbf{b}\| = \|\mathbf{b}\||\cos\theta| = \frac{\|\mathbf{a}\|}{\|\mathbf{a}\|}\|\mathbf{b}\||\cos\theta| = \frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}$$

Thus, we know the magnitude and direction of $\mathsf{proj}_{\mathbf{a}}\mathbf{b}$

We know:

1. The direction of the projection is $\pm\mathbf{a}$. A unit vector on this direction is $\pm\frac{\mathbf{a}}{\|\mathbf{a}\|}$.
2. Has norm $\frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}$.

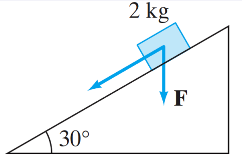
So the **projection vector** $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$ is:

Formula for $\mathrm{proj}_{\mathbf{a}}\mathbf{b}$

$$\mathrm{proj}_{\mathbf{a}}\mathbf{b} = \pm\left(\frac{|\mathbf{a}\cdot\mathbf{b}|}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \pm\left(\frac{\pm\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|}\right)\frac{\mathbf{a}}{\|\mathbf{a}\|} = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|^2}\mathbf{a}$$

Example 4

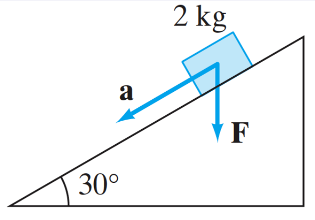
- Suppose that a 2 kg object is sliding down a ramp
- The ramp has a $30°$ incline with the horizontal



- If we neglect friction, the only force acting on the object is gravity

What is the component of the gravitational force in the direction of motion of the object?

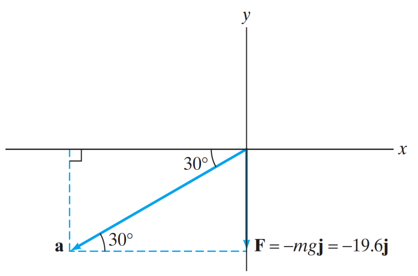
Example 4



- We need to calculate $\text{proj}_{\mathbf{a}}\mathbf{F}$
- $\mathbf{F}$ is the gravitational force vector
- $\mathbf{a}$ points along the ramp as shown in figure.

Example 4

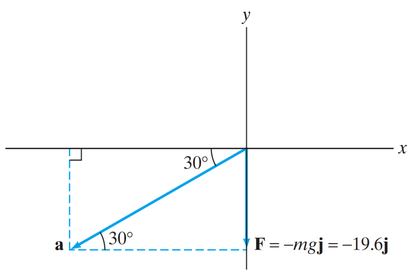
- The coordinate situation is shown in figure



- The vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$ has the form,

$$a_1 = \|\mathbf{a}\| \cos 210^\circ \text{ and } a_2 = \|\mathbf{a}\| \sin 210^\circ$$

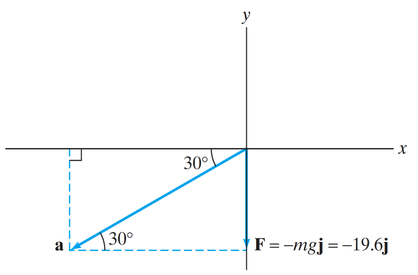
Example 4



- We are really only interested in the direction of $\mathbf{a}$, because the projection will be the same for any length of a.
- There is no loss in assuming that $\mathbf{a}$ is a **unit vector**.

$$\mathbf{a} = (\cos 210^\circ, \sin 210^\circ) = -\cos 30^\circ \mathbf{i} - \sin 30^\circ \mathbf{j} = -\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}$$

Example 4



- Taking $g = 9.8\text{m}/\text{sec}^2$, we have $\mathbf{F} = -mg = -2g\mathbf{j} = -19.6\mathbf{j}$
- Therefore,

$$\text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{\mathbf{a}\cdot\mathbf{F}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)\cdot(-19.6\mathbf{j})}{1}\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$$

Example 4

$$\text{proj}_{\mathbf{a}}\mathbf{F} = \left(\frac{\mathbf{a}\cdot\mathbf{F}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a} = \frac{\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)\cdot(-19.6\mathbf{j})}{1}\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right)$$

$$= 9.8\left(-\frac{\sqrt{3}}{2}\mathbf{i} - \frac{1}{2}\mathbf{j}\right) \approx -8.49\mathbf{i} - 4.9\mathbf{j}$$

- And the component of $\mathbf{F}$ in this direction is

$$\|\text{proj}_{\mathbf{a}}\mathbf{F}\| = \|-8.49\mathbf{i} - 4.9\mathbf{j}\| = 9.8 \text{ N}$$

Normalization of a vector

- Unit vectors, that is, vectors of length 1, are important in that they capture the idea of direction

They all have the same length

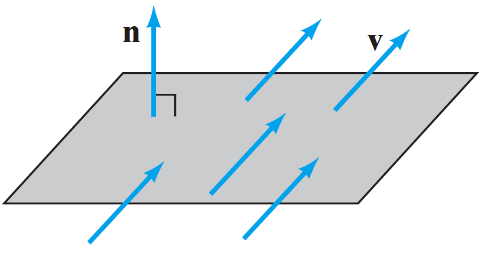
- **Proposition 3.4** shows that every nonzero vector $\mathbf{a}$ can have its length adjusted to give a unit vector

$$\mathbf{u} = \frac{\mathbf{a}}{\|\mathbf{a}\|}$$

- $\mathbf{u}$ points in the same direction as $\mathbf{a}$.
- This operation is referred to as **normalization** of the vector. $\mathbf{a}$

Example 5

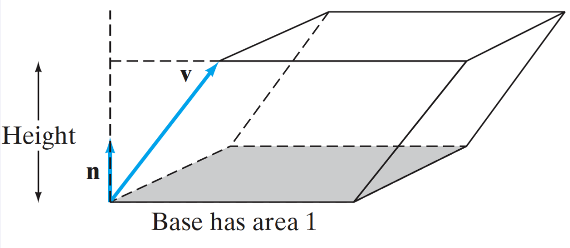
- A fluid is flowing across a plane surface with uniform velocity $\mathbf{v}$.
- Let $\mathbf{n}$ be a unit vector perpendicular to the plane surface:



- Find (in terms of $\mathbf{v}$ and $\mathbf{n}$) the volume of the fluid that passes through a unit area of the plane in unit time.

Example 5

- Suppose one unit of time has elapsed, v = space/time = space, for time=1.
- Then, over a unit area of the plane (a unit square), the fluid will have filled a "box" as in figure.



- The box may be represented by a **parallelepiped**.
- The volume we seek is the volume of this parallelepiped.

Example 5

- The volume of this parallelepiped is:

$$\text{Volume} = (\text{area of base})\,(\text{height})$$

- The area of the base is 1 unit by construction.
- The height is given by $\text{proj}_{\mathbf{n}}\mathbf{v}$.
- Since $\mathbf{n} \cdot \mathbf{n} = \|\mathbf{n}\|^2 = 1$

$$\text{proj}_{\mathbf{n}}\mathbf{v} \;\; = \;\; \left(\frac{\mathbf{n} \cdot \mathbf{v}}{\mathbf{n} \cdot \mathbf{n}}\right)\mathbf{n} = (\mathbf{n} \cdot \mathbf{v})\mathbf{n}$$

- Hence

$$\|\text{proj}_{\mathbf{n}}\mathbf{v}\| = \|(\mathbf{n} \cdot \mathbf{v})\mathbf{n}\| = |\mathbf{n} \cdot \mathbf{v}|\|\mathbf{n}\| = |\mathbf{n} \cdot \mathbf{v}|$$

Outline

Motivation

- The **cross product** of two vectors in $\mathbb{R}^3$ is an "honest" product,

  it takes two vectors
  and produces a third one

- However, the cross product possesses less "natural" properties:

  it cannot be defined for vectors in $\mathbb{R}^2$
  without first embedding them in $\mathbb{R}^3$

- Intuitively the **cross product** of two vectors gives another vector perpendicular to both of them. It has norm $\|\mathbf{a}\|\|\mathbf{b}\||\sin\theta|$, the area of the parallelogram formed by the vector **a** and **b**.

To introduces the definition of cross product we need to remember some **Matrix Algebra**.

Matrices

- A **matrix** is a rectangular array of numbers.
- Examples of matrices are

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 7 \\ 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- If a matrix has n rows and m columns, we write it $n \times m$.
- Thus, the three matrices just mentioned are, respectively, 2×3, 3×2 and 4×4.
- To some extent, matrices behave algebraically like vectors.
- Mainly interesting for us is the the notion of a **determinant**.
- It is a real number associated to an **square** matrix $n \times n$.

Definition 4.2: Determinants

- Let A be a 2×2 or 3×3 matrix.
- Then the **determinant** of A, denoted **det A** or $|A|$, is the real number computed from the individual entries of A as follows:

1. 2×2 case

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Definition 4.2: Determinants

2. 3×3 case

If,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

then,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - afh - bdi$$

Definition 4.2: Determinants

3. 3×3 case in terms of 2×2 determinants

If,

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

then,

$$|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

In this case we develop the matrix by **minors**. This is the general form to calculate a determinant for an arbitrary square matrix A.

There are mnemonic rules for this

Diagonal Approach for 2×2 and 3×3 Determinants

- We write (or imagine) diagonal lines running through the matrix entries

It is not valid
for higher-order determinants

1. 2×2 case

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$$|A| = ad - bc$$

Diagonal Approach for 2×2 and 3×3 Determinants

2. 3×3 case

We need to repeat the first two columns
for the method to work

$$\begin{array}{l} \qquad\qquad\qquad\qquad\qquad - \quad - \quad - \\ A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{matrix} a & b \\ d & e \\ g & h \end{matrix} \\ \qquad\qquad\qquad\qquad\qquad + \quad + \quad + \end{array}$$

$$|A| = aei + bfg + cdh - ceg - afh - bdi$$

Definition of Cross Product

The **cross product** of two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}\mathbf{k}$$

Example 3

$$(3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \times (\mathbf{i} - \mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \mathbf{i} - 4\mathbf{j} - 5\mathbf{k}$$

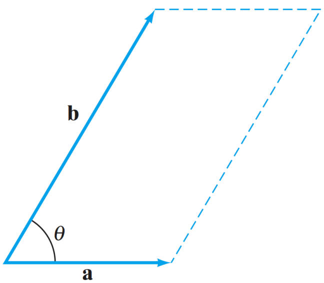
Properties

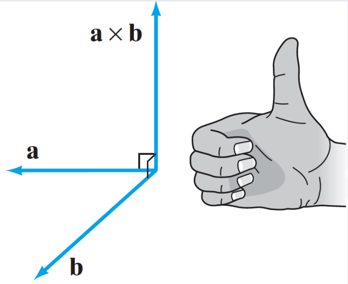
- The **direction** of $\mathbf{a} \times \mathbf{b}$ is such that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ (when both $\mathbf{a}$ and $\mathbf{b}$ are nonzero). v
- It is taken so that the ordered triple $(\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b})$ is a right-handed set of vectors.
- The **length** of $\mathbf{a} \times \mathbf{b}$ is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$ or is zero if either $\mathbf{a}$ is parallel to $\mathbf{b}$ or if $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$.
- Alternatively, the following formula holds

$$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

The norm and orientation of the cross product





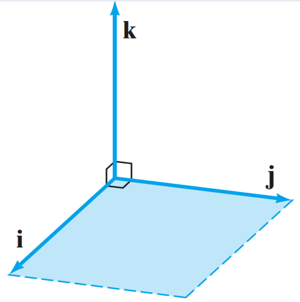
- The area of this parallelogram is,

$$\|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$$

Example

- Compute the cross product of the standard basis vectors for $\mathbb{R}^3$

- First consider $\mathbf{i} \times \mathbf{j}$ as shown in figure



- The vectors $\mathbf{i}$ and $\mathbf{j}$ determine a square of unit area.

Example

- Compute the cross product of the standard basis vectors for $\mathbb{R}^3$

- The vectors $\mathbf{i}$ and $\mathbf{j}$ determine a square of unit area
- Thus,
$$\|\mathbf{i} \times \mathbf{j}\| = 1$$
- Any vector perpendicular to both $\mathbf{i}$ and $\mathbf{j}$ must be perpendicular to the plane in which $\mathbf{i}$ and $\mathbf{j}$ lie.
- Hence, $\mathbf{i} \times \mathbf{j}$ must point in the direction of $\pm k$
- The **right-hand rule** implies that $\mathbf{i} \times \mathbf{j}$ must point in the positive k direction
- Since $\|\mathbf{k}\| = 1$, we conclude that,
$$\mathbf{i} \times \mathbf{j} = \mathbf{k}$$

Properties of the Cross Product

- Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$ and let $k \in \mathbb{R}$ be any scalar. Then:
  1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ **(anticommutativity)**
  2. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ **(distributivity)**
  3. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ **(distributivity)**
  4. $k(\mathbf{a} \times \mathbf{b}) = (k\mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (k\mathbf{b})$ **(associative with scalars)**

It is not associative with vectors as we'll see in the next slide.

Properties the Cross Product Does Not Fulfil

- Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be vectors in $\mathbb{R}^3$ and let $k \in \mathbb{R}$ be any scalar.
- In general, the cross product is not commutative

$$\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$$

- In general, the cross product does not fulfill associativity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$$

Example

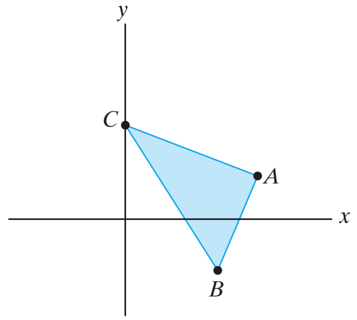
Let $\mathbf{a} = \mathbf{b} = \mathbf{i}$ and $\mathbf{c} = \mathbf{j}$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{k} \times \mathbf{i} = -\mathbf{j}$$

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{j} = \mathbf{0}$$

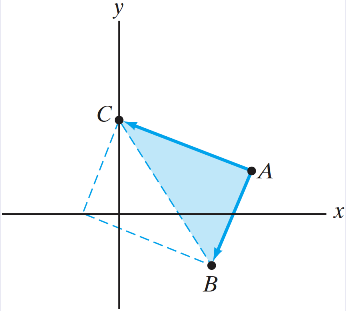
Example

Use vectors to calculate the area of the triangle whose vertices are $A(3,1), B(2,-1)$, and $C(0,2)$ as shown in figure:



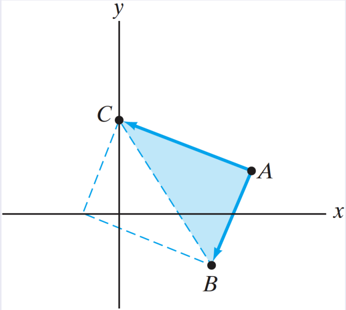
Example

- The trick is to recognise that any triangle can be thought of as half of a parallelogram,



- Now, the area of a parallelogram is obtained from a cross product.

Example



- $\overrightarrow{AB} \times \overrightarrow{AC}$ is a vector whose length measures the area of the parallelogram determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$

$$\text{Area of } \nabla ABC = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

Example

- To use the cross product, we must consider $\overrightarrow{AB}, \overrightarrow{AC} \in \mathbb{R}^3$
- We simply take the k-components to be zero

$$\begin{aligned}\overrightarrow{AB} &= -\mathbf{i} - 2\mathbf{j} = -\mathbf{i} - 2\mathbf{j} - 0\mathbf{k} \\ \overrightarrow{AC} &= -3\mathbf{i} + \mathbf{j} = -3\mathbf{i} + \mathbf{j} + 0\mathbf{k}\end{aligned}$$

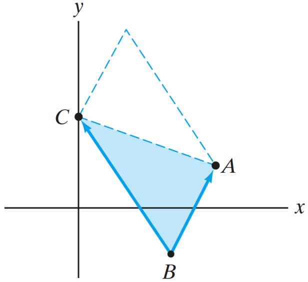
- Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ -3 & 1 & 0 \end{vmatrix} = -7\mathbf{k}$$

$$\text{Area of } \nabla ABC = \frac{1}{2}\|-7\mathbf{k}\| = \frac{7}{2}$$

Example

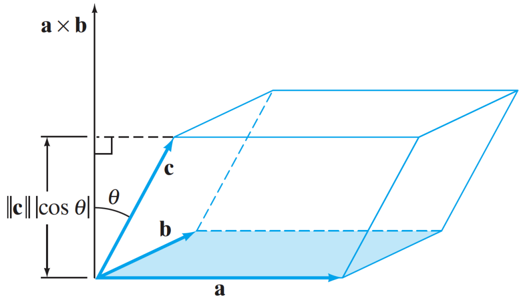
- There is nothing sacred about using A as the common vertex
- We could just as easily have used B or C, as shown in figure



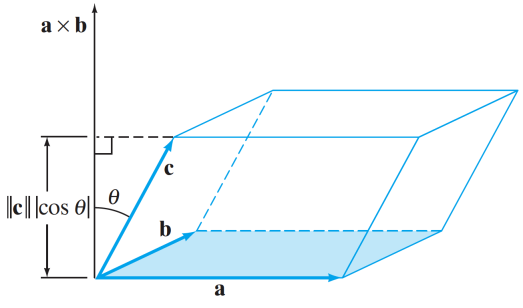
$$\begin{aligned}\text{Area of } \nabla ABC &= \frac{1}{2}\|\overrightarrow{BA} \times \overrightarrow{BC}\| = \frac{1}{2}\|(\mathbf{i} + 2\mathbf{j}) \times (-2\mathbf{i} + 3\mathbf{j})\| \\ &= \frac{1}{2}\|7\mathbf{k}\| = \frac{7}{2}\end{aligned}$$

Example

Find a formula for the volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$:

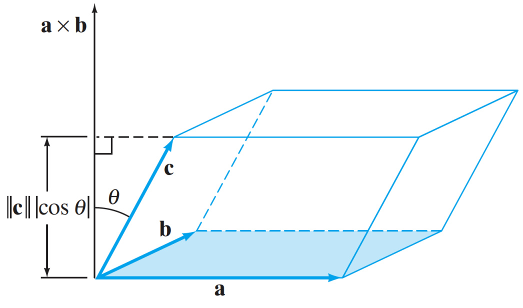


Example



- The volume of a parallelepiped is equal to the product of the area of the base and the height.
- The base is the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.
- Its area is $\|\mathbf{a} \times \mathbf{b}\|$.

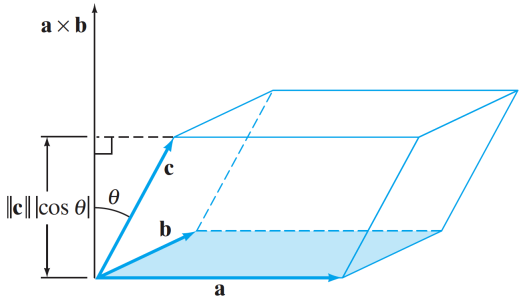
Example



- The vector $\mathbf{a} \times \mathbf{b}$ is perpendicular to this parallelogram.
- The height of the parallelepiped is $\|\mathbf{c}\| |\cos\theta|$.
- θ is the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$.

The absolute value is needed in case $\theta > \frac{\pi}{2}$

Example



Volume of parallelepiped = (area of base)(height)

$$= \|\mathbf{a}\times\mathbf{b}\|\|\mathbf{c}\||\cos\theta| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$$

Example

Volume of parallelepiped =

$$\text{(area of base)(height)}$$
$$= \|\mathbf{a}\times\mathbf{b}\|\|\mathbf{c}\||\cos\theta| = |(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|$$

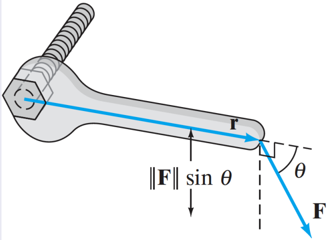
For example, the parallelepiped determined by the vectors

$$\mathbf{a} = \mathbf{i} + 5\mathbf{j}, \;\; \mathbf{b} = -4\mathbf{i} + 2\mathbf{j} \;\text{ and }\; \mathbf{c} = \mathbf{i} + \mathbf{j} + 6\mathbf{k}$$

$$\begin{aligned}\text{Volume of parallelepiped} &= |((\mathbf{i}+5\mathbf{j})\times(-4\mathbf{i}+2\mathbf{j}))\cdot(\mathbf{i}+\mathbf{j}+6\mathbf{k})| \\ &= |22\mathbf{k}\cdot(\mathbf{i}+\mathbf{j}+6\mathbf{k})| = |22(6)| = 132\end{aligned}$$

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt:

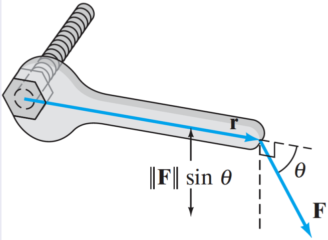


- To measure exactly how much the bolt moves, we need the notion of **torque** (or **twisting force**).
- Letting F denote the force you apply to the wrench. Then:

Amount of torque = (wrench length)(component of F ⊥ wrench)

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



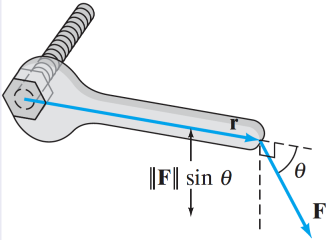
- Let $\mathbf{r}$ be the vector from the center of the bolt head to the end of the wrench handle
- Then

$$\text{Amount of torque} = \|\mathbf{r}\|\|\mathbf{F}\|\sin\theta$$

where θ is the angle between $\mathbf{r}$ and $\mathbf{F}$.

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt



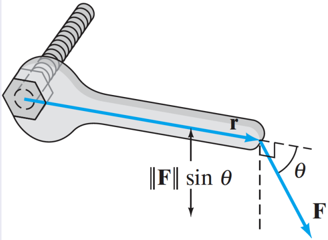
- That is, the amount of torque is

$$\|\mathbf{r} \times \mathbf{F}\|$$

- And the direction of $\mathbf{r} \times \mathbf{F}$ is the same as the direction in which the bolt moves.

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt

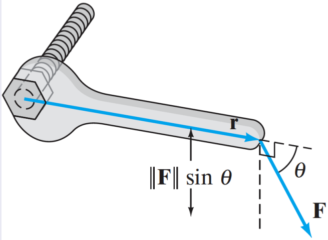


- Hence, it is quite natural to define the **torque vector** $\mathbf{T}$ to be

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}$$

Turning a bolt with a wrench

- Suppose you use a wrench to turn a bolt

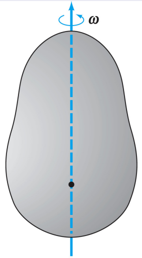


- Note that if $\mathbf{F}$ is parallel to $\mathbf{r}$, then $\mathbf{T} = \mathbf{0}$

If you try to push or pull the wrench,
the bolt does not turn

Spinning an object about an axis

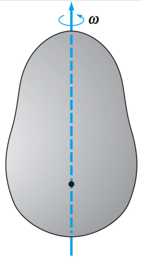
- Assume the rotation of a rigid body about an axis as shown in figure



What is the relation between
the (linear) velocity of a point of the object
and the rotational velocity?

Spinning an object about an axis

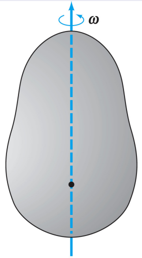
- Assume the rotation of a rigid body about an axis as shown in figure



- First, we need to define a vector ω, the **angular velocity vector** of the rotation
- This vector points along the axis of rotation, and its direction is determined by the right-hand rule

Spinning an object about an axis

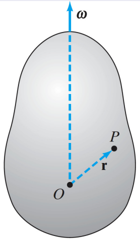
- Assume the rotation of a rigid body about an axis as shown in figure



- The magnitude of ω is the angular speed (measured in radians per unit time) at which the object spins
- Assume that the angular speed is constant in this discussion

Spinning an object about an axis

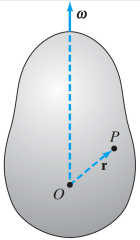
- Assume the rotation of a rigid body about an axis as shown in figure



- Fix a point O (the origin) on the axis of rotation
- Let $\mathbf{r}(t) = \overrightarrow{OP}$ be the position vector of a point P of the body, measured as a function of time

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

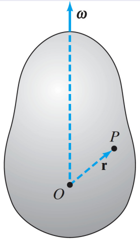


- The velocity $\mathbf{v}$ of P is defined by

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}$$

Spinning an object about an axis

- Assume the rotation of a rigid body about an axis as shown in figure

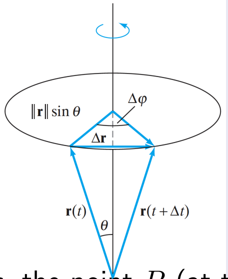


- $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$

The vector change in position between times t and $t + \Delta t$

- Our goal is to relate $\mathbf{v}$ and ω

Spinning an object about an axis

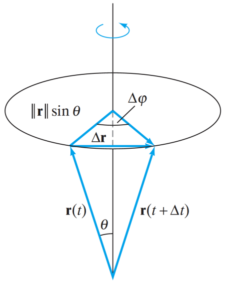


- As the body rotates, the point P (at the tip of the vector $\mathbf{r}$) moves in a circle whose plane is perpendicular to ω
- The radius of this circle is

$$\|\mathbf{r}(t)\|\sin\theta$$

where θ is the angle between ω and $\mathbf{r}$

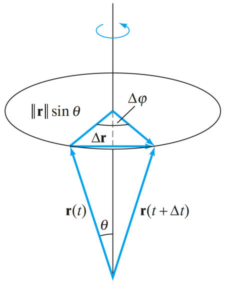
Spinning an object about an axis



- Both $\|\mathbf{r}(t)\|$ and θ must be constant for this rotation

The direction of $\mathbf{r}(t)$
may change with t, however

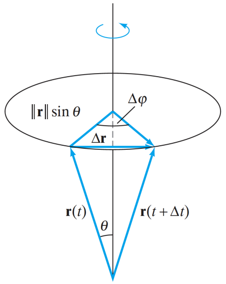
Spinning an object about an axis



- If $t \approx 0$, then $\|\Delta\mathbf{r}\|$ is approximately the length of the circular arc swept by P between t and $t + \Delta t$
- That is,

$$\begin{aligned} \|\Delta\mathbf{r}\| &\approx \text{(radius of circle)(angle swept through by } P\text{)} \\ &= (\|\mathbf{r}\|\sin\theta)(\Delta\phi) \end{aligned}$$

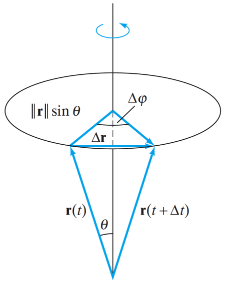
Spinning an object about an axis



- Thus

$$\left\| \frac{\Delta \mathbf{r}}{\Delta t} \right\| \approx \|\mathbf{r}\| \sin\theta \frac{\Delta \phi}{\Delta t}$$

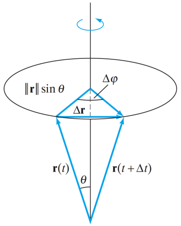
Spinning an object about an axis



- Now, let $\Delta t \to 0$
- Then $\frac{\Delta \mathbf{r}}{\Delta t} \to \mathbf{v}$ and $\frac{\Delta \phi}{\Delta t} \to \|\omega\|$ by definition of the angular velocity vector ω
- Thus, we have

$$\|\mathbf{v}\| = \|\omega\| \|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|$$

Spinning an object about an axis



$$\|\mathbf{v}\| = \|\omega\| \|\mathbf{r}\| \sin\theta = \|\omega \times \mathbf{r}\|$$

- It's not difficult to see intuitively that $\mathbf{v}$ must be perpendicular to both ω and $\mathbf{r}$
- Right-hand rule should enable you to establish the vector equation

$$\mathbf{v} = \omega \times \mathbf{r}$$

Spinning an object about an axis

- Apply to a bicycle wheel formula

$$\|\mathbf{v}\| = \|\omega\|\|\mathbf{r}\|\sin\theta = \|\omega \times \mathbf{r}\|$$

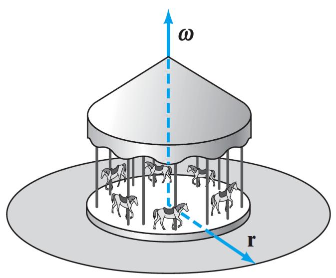
- It tells us that the speed of a point on the edge of the wheel is equal to the product of
    - The radius of the wheel, and
    - The angular speed

θ is $\frac{\pi}{2}$ in this case

- If the rate of rotation is kept constant, a point on the rim of a large wheel goes faster than a point on the rim of a small one

Spinning an object about an axis

- In the case of a carousel wheel, this result tells you to sit on an outside horse if you want a more exciting ride.



Outline

Here we resume the properties:

Scalar Multiplication: $k\mathbf{a}$

- Result is a vector in the direction of $\mathbf{a}$
- Magnitude is $\|k\mathbf{a}\| = |k|\|\mathbf{a}\|$
- Zero if $k = 0$ or $\mathbf{a} = \mathbf{0}$
- Commutative: $k\mathbf{a} = \mathbf{a}k$
- Associative: $k(l\mathbf{a}) = (kl)\mathbf{a}$
- Distributive: $k(\mathbf{a} + \mathbf{b}) = k\mathbf{a} + k\mathbf{b}$ and $(k + l)\mathbf{a} = k\mathbf{a} + l\mathbf{a}$

Dot Product: $\mathbf{a} \cdot \mathbf{b}$

- Result is a scalar
- Magnitude is $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\| \cos\theta$; θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is maximized if $\mathbf{a} \parallel \mathbf{b}$
- Zero if $\mathbf{a} \perp \mathbf{b}$, $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
- Commutative: $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- Associativity is irrelevant, since $(\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{c}$ doesn't make sense
- Distributive: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
- If $\mathbf{a} = \mathbf{b}$ then $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$

Cross Product: $\mathbf{a} \times \mathbf{b}$

- Result is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta$; θ is the angle between $\mathbf{a}$ and $\mathbf{b}$
- Magnitude is maximized if $\mathbf{a} \perp \mathbf{b}$
- Zero if $\mathbf{a} \parallel \mathbf{b}$, $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$
- Anticommutative: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
- Not associative: In general $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \neq (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- Distributive: $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ and $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
- If $\mathbf{a} \perp \mathbf{b}$ then $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|$