Class 9

Cash Discounts

Cash Discounts

- When buying a product you may be offered a choice between two methods of payments
  - cash "price"
  - delayed payment
- The Cash price is the total (net) amount of money you have to pay when you make the purchase and complete the transaction immediately
- The delayed payment is the set of dated payments you agree to pay in exchange for taking the purchased good home immediately

Simple example

- Buy a TV and pay 1200€ today (cash) or pay 1300€ in 6 months time. Compute the effective APR

$$1200 = \frac{1300}{(1+r)^{1/2}} \Rightarrow r = 17.361\%$$

Typical example

- Buy a truckload of soft drinks with a price tag of 1200€. You are offered two options
  - pay cash and receive a 5% cash discount
  - pay 1200€ in 120 days

  Compute the effective APR of the transaction using 30:360
- Answer
  - If you pay cash you pay $1200 - 5$€
  - If you wait 120 days (4/12 of a year) then you pay 1200€, then

$$1140 = \frac{1200}{(1+r)^{4/12}} \Rightarrow r = 16.635\%$$

Short-term Operations and the Effective APR

Commercial characteristics

- In financial transactions there are additional characteristics that can substantially modify the set of dated payments to make and/or receive, which alter the computation of financial equivalence between provision and compensation, and hence the cost/return of the transaction.
- When these characteristics are incorporated into the provision and compensation, the resulting dated payments are called the real provision and real compensation.
- Commercial characteristics can be classified into two groups:
  - bilateral characteristics
  - unilateral characteristics

Bilateral characteristics

- These are the ones that affect both parties, the borrower and the lender
- Thus, the party that receives the provision has makes or receives a payment that is received or made by the counterparty.
- Example: initial commission, discounts, premia, charges, ...

Unilateral characteristics

- These characteristics affect only one of the two parties involved in the transaction. These payments usually arise with the participation of a third party from outside the transaction.
- Example: notary payments, taxes, etc
- As one of the parties has to increase the payments made but this does not translate in an increase in the payments received by the counterparty, there is a need to differentiate between real provision and real compensation

Real effective rate

- Once the commercial characteristics of the transaction have been introduced, we will now define the effective real interest rate. This is the effective APR but it includes commercial characteristics, and it is computed using annual compounding.
- If there are no commercial characteristics and uses the compound law, the real effective rate coincides with the Effective APR defined previously.
- In case all commercial characteristics are bilateral, then the real effective rate is the one that financially equates the real provision with the real compensation. Evidently, this calculation is the same for the borrower as for the lender. Only this interest rate represents the return to the lender and the cost for the borrower.
- If the characteristics are unilateral then we differentiate the real effective rate for the borrower and the real effective rate for the lender, both of which are obtained by equating the real provision and real compensation for each of the two parties.

Real effective rates

- Thus, one defines the real effective rate for the borrower as the annual interest rate calculated with annual compounding that generates financial equivalent between **the borrower's real provision and real compensation**.
- And, one defines the real effective rate for the lender as the annual interest rate calculated with annual compounding that generates financial equivalent between **the lender's real provision and real compensation**.

Example

- A bank offers a savings options that is as follows: deposit 14.000 € and in two and one half years you will receive 15000€.
- The lender (us) has initial costs of 0.4% on the nominal amount deposited that go to the borrower, and a final payment of 0.2% on the initial nominal amount for a third party.
- Compute
  - The effective interest rate for the operation without the additional commercial characteristics
  - The real effective interest for the borrower
  - The real effective interest for the lender

Solution

- The "pure" transaction–without commercial characteristics

$$14000\,(1+r)^{2.5} = 15000 \Rightarrow r = 2.80\%$$

- The real effective interest for the borrower (bank): receives $14.000 + 0,4\% = 14056$

$$14056\,(1+r)^{2.5} = 15000 \Rightarrow r = 2.63\%$$

- The real effective interest for the lender (us): deposit $14.000 + 0,4\% = 14056$ and receives $15.000 - 0,2\% = 14972$

$$14056\,(1+r)^{2.5} = 15000 - 28 \Rightarrow r = 2.56\%$$

Calculating the Effective APR (TAE) according to Bank of Spain rules

- A financial transaction is an agreement between two parties, and in principle, any agreement that does not go against the general laws is valid. This implies that transactions between private parties are not tied to any specific regulation and hence the computation of the cost or return can be done in any way they consider valid.
- But, when one of the parties is a *financial entity*, there is specific regulation that specifies how to calculate the cost and return of the operation with the objective of providing transparency to the market and more information to clients. These are:
  - Circulares del Banco de España nº 8 de 7 de Septiembre de 1990 y nº 13 de 21 de Diciembre de 1993

Bank of Spain rules

Basically, the Bank of Spain imposes that:

- Advertisement of interest rates that will be applied on transactions
- Provision of information on commissions and costs the client has to face
- As for the effective cost or return of transactions:
  - the computation of the Effective APR (TAE) will not include unilateral characteristics
  - the effective rate computed is the one for the financial firm, not the client
  - for the client this information is only designed to be helpful

Example

We request a 1000€ loan to be returned in 12 monthly installments. The commission to process the loan is 18€. What is the transaction's effective APR?

Example

- Nominal interest rate is 0%
- You will pay $1000/12$ = 83,33€ each month
- You will receive 982€ = 1000€ - 18€

$$\begin{aligned} 982 &= \frac{83,33}{(1+r)^{1/12}} + \frac{83,33}{(1+r)^{2/12}} + \ldots + \frac{83,33}{(1+r)^{12/12}} \\ r &= 3.41\% \end{aligned}$$

Exercise

- A financial entity has given us a loan of 60.000€ at an 8% annual interest rate with annual compounding which we have to pay back in two years and a half's time.
- Compute the amount of money to be returned two years and a half from today.

$$60000\,(1+8\%)^{2.5} = 72,729.51$$

- The initial costs of the loan are 380€ which will be paid to the financial entity. In case you decide to cancel early, the entity will penalize us with 3% of the remaining balance on the loan.
- Compute the effective real interest rate for the client and the effective rate for the financial entity

$$\begin{aligned} (60000-380)\,(1+r)^{2.5} &= 72,729.51 \\ r &= 8.27\% \end{aligned}$$

Exercise

- A year and a half since the loan was accepted, the conditions for loans have changed substantially. We find another financial entity Y which offers a 6% annual interest rate with monthly capitalization, plus an initial 0.25% commission on the loans' principal.
- We decide to cancel the first loan and request the one from entity Y for the necessary amount.
- Compute:
  - The amount of money that we request from Y
  - Amount to be paid back to Y at the (original) end of the loan
  - Effective rate of the new loan.
  - Did you make the right decision to change loans?

Solution

- Amount due: $60000\,(1+8\%)^{1.5} = 67,342.14$
  - Penalty: $3\% * 67,342.14 = 2,020.26$
  - Total requested: $67,342.14 + 2,020.26 = 69,362.40$€
  - Total requested (including commission): $69,362.40/(1-0.25\%) = 69,536.24$€
- Final payment: $69,536.24\,(1+6\%/12)^{12} = 73,825.08$€
- Effective rate: provision $69,536.24\,(1-0.25\%) = 69,362.40$€, compensation: $73,825.08$€

$$69,362.40\,(1+r) = 73,825.08 \Rightarrow r = 6.43\%$$

- Was it a good decision? NO
  - 8% loan: final payment = $72,729.51$€
  - change of loan: final payment = $73,825.08$€

Exercise

- We want to install new antivirus software on all the university's computers. The software company offers us two forms of payment for the annual license:
  - 20.000€ cash or
  - two semestral payments of 11.000€ each, the first at the start of the contract and the second 6 months later
- What is the effective APR of the loan implied in the choice of form of payment?

Solution

- Provision: $20,000$
- Compensation: $(11,000;0)$ and $(11,000;6m)$

$$\begin{aligned} 20000 &= 11000 + 1000\,(1+r)^{1/2} \\ r &= 49.38\% \end{aligned}$$

Exercise

We want to install an antivirus on all the University's computers. The company that maintains the hardware makes two offers:

- 20.000€ in cash upfront
- Two payments of 11.000€ each, one in cash and the other 6 months later

What is the real effective rate (TAE) of this financial transaction?

Solution

- Provision: option 1: 20.000€ today
- Compensation: option 2:
  - 11.000€ today
  - 11.000€ in 6 months

We can calculate the net exchange of money

- receive 9.000€ (20.000-11.000) today in exchange for paying 11.000 in 6 months

$$\begin{aligned} 1 + r_{6mon} &= \frac{11000}{9000} \\ r &= (1 + r_{6mon})^2 - 1 = 49\%. \end{aligned}$$

Exercise

You take three IOUs to the bank, to get cash. They are

1. Nominal: 3,000€, due in 30 days
2. Nominal: 5,500€, due in 45 days
3. Nominal: 1,500€, due in 80 days

The bank's discounting terms are: 10% annual, 0,3% of the nominal amount as commission (payable immediately)
Determine:

- The amount of cash the company receives
- The effective interest rate for the company (a) using the single equivalent IOU (b) using the three payments (and the Excel function Solver)
- Government bonds
- Corporate bonds
- Repos & interbank deposits

Class 11

Letras del Tesoro

Short-term Financial Instruments

- So far, we have worked with financial transactions (investments and loans) that are typical between a street bank and its customers
- In this class we will look at other financial instruments used for short-term investments and financing
- These instruments are those traded in financial markets: fixed income markets and money markets

Monetary financial markets (the money market)

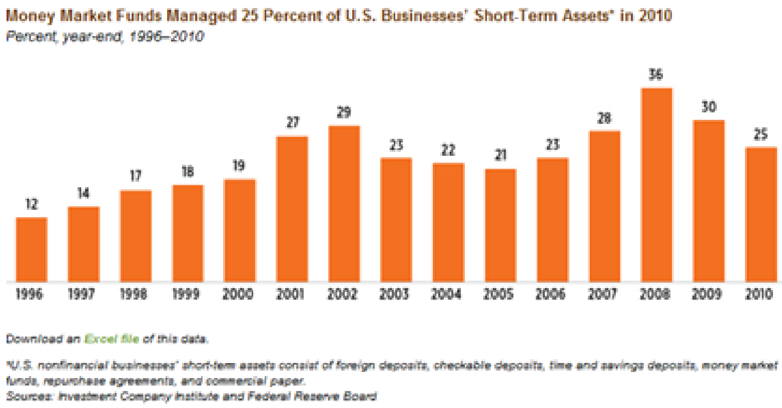
- Money markets trade debt instruments with short maturity, low risk, and high liquidity. These are (primarily) Government bonds, corporate bonds, repos & interbank deposits
  - Short maturity: the asset disappears in less than (around) one year to 18 months
  - Low risk: this comes from the short maturity, the credit quality of the issuers of the debt, and the guarantees (in the form of real assets) attached to these assets
  - High liquidity: obtained from the existence of active organized secondary markets which facilitate a quick and easy liquidation of the asset

Money Markets

Issuer	Market	Instruments
Tesoro Público	Short-term government debt	Letras del Tesoro
Non-financial firms	Short-term corporate debt	Pagarés de empresa
Banks and savings banks	Interbank Money Markets	Interbank deposits Pagarés bancarios

Investing in Money Markets

- Investing in short-term instruments is associated with low risk & low return. Can seem unattractive
- Determining the bankruptcy risks of the issuer may not compensate for the return offered
- Most investors purchase these products through money market mutual funds (FIAMM), which has greatly enhanced the market for money market instruments
- These funds include many debt instruments from the money market which allows for risk diversification
- In 2008, 36% of all short-term instruments where held by FIAMM funds
- Other investors in these markets: large banks, pension funds, mutual funds, etc



The market for short-term government debt (which is a subset of the money market) is itself a segment of the market for assets issued by El Tesoro. This latter market includes other government (long-term) debt.

The government's issues of short term debt have not always been the same, and have evolved over time, the first regular issues (of Spanish debt) starting around 1973.

1973-82	Bonos del tesoro, del Ministerio de Hacienda
1980-82	Certificados de deposito emitidos por el banco de España
1982-83	Certificados de regulación monetaria
1981-84	Pagarés del tesoro
1987-hoy	**Letras del tesoro**

Buying Public Debt

Primary Market
- All public debt is issued through a competitive public auction
- Anyone can participate in the auction via a "Entidad gestora"

Secondary Market
- All public debt is traded in the secondary market (www.AIAF.es)
- An investor can then do so with horizons that differ from those fixed by the Tesoro

Letras del Tesoro

- These are short term fixed income securities. They have been issued by el Tesoro since 1987 and exist solely as bank annotations.
- These are issued via an auction, which is run by the Bank of Spain once every two weeks. The minimum amount you can bid is 1.000€, and requests above this amount have to be multiples of 1.000
- Currently, el Tesoro issues Letras del Tesoro with maturities: 3, 6, 12 and 18 months
- These are securities issued at a discount. This means that their price will be below their nominal amount, which is what the investor will receive at maturity. The difference between the **nominal** amount of the Letra (1.000€) and its purchase price will be the interest generated by the Letra del Tesoro.

Letras del Tesoro

- Their returns are subject to IRPF (income tax) or IS (corporate tax), but are not subject to withholding.
- In order to compute the price of a Letra you have to take into account the duration of the operation, as the financial law to be applied is different in the following two cases though the day count convention is always (actual:360):
  - Simple capitalization if the duration is less than (or equal) a natural year
  - Compound capitalization if it is greater than a natural year
- These assets have full liquidity and guarantee, the investor can wait for the maturity date or sell them in the secondary market at any time prior to its maturity
- In the latter case, the return is determined by the market price at the time of sale, so that the investor does not know, at the time of purchase, what the final return on his investment will be.

Letras del Tesoro

- Imagine you purchase a 12 month Letra del Tesoro at the last auction and you paid a price of 958,50€ (you will receive 1.000€ at maturity).
  - What is the return on this investment if you hold it until maturity?

$$\begin{aligned} 958,50\,(1+r) &= 1.000 \\ r &= 4,3\% \end{aligned}$$

  - What if the Letra had been a 6 month maturity one?

$$\begin{aligned} 958,50\,(1+r)^{\frac{180}{360}} &= 1.000 \\ r &= 8,847\% \end{aligned}$$

Example

On July 14, 2010, Letras were issued till July 13, 2011 (364 days). The price at auction was 95.7665%.

- What is the official return on this asset?

$$\begin{aligned} 95.7665\% \left(1 + r\frac{364}{360}\right) &= 1000 \\ r = 0.0437 &\Rightarrow 4.37\%. \end{aligned}$$

What if they had expired in Jan 12, 2012?

$$\begin{aligned} 95.7665\,(1+r)^{548/360} &= 1000 \\ r = 0.0288 &\Rightarrow 2.88\%. \end{aligned}$$

If expiry is less than or equal to one natural year => simple capitalization, otherwise compound.

Example

You purchase a Letra on the market on the 1st of oct, 2008. It matures on the 1st of april, 2009, and you paid 989.56 euros. The costs that you paid to make the purchase are:

- Gastos de intervención: 0.05% on the nominal value, paid at the time of purchase
- A 0.1% commission for the administrative costs, paid at maturity and based on the nominal value of the asset

Compute the return obtained from this investment.

Letras del Tesoro Auctions

Letras del Tesoro

- The primary issue of Letras del Tesoro is done via auction
- Trading occurs in the secondary market, specifically via bank annotations. Since 1987 there do not exist any physical security. Trading occurs in the SENAF exchange.
- Bank annotations are a specific form of security whereby a security is identified via entries on a special accounting registry, usually a computerized one.

Letras del Tesoro: Auctions

- El Tesoro issues these securities via a competitive auction system (Letras, bonos and obligaciones).
- At the beginning of each year, el Tesoro publishes a full calendar with the dates in which ordinary auctions will take place. Outside of this calendar, they can also run special auctions.
- You can make two types of bids/offers at these auctions
  - Competitive bids: you announce the nominal value of the securities you wish to purchase and at what price you are willing to purchase them (the latter as a % of the nominal value)
  - Non-competitive bids: you only announce the nominal value of your desired purchases. These are the more adequate ones for the small investor, as through these he guarantees that his request will be fulfilled.

The auction

- Once all requests and bids have been received and the deadline reached, competitive bids are ordered from highest to lowest price, and from this, the bank determines: the nominal (or effective) amount to be issued and the minimum price (marginal price) accepted at auction
- This price is expressed as a percentage rate with three decimal digits
- All non competitive bids plus all competitive bids with a price equal to or greater than the minimum accepted amount will be automatically accepted, and all the other bids will be rejected. There exists the possibility that those offers made at the marginal rate may be prorated.
- From the competitive bids, one determines the weighted average price (precio medio ponderado, PMP). This is an average that weighs prices in terms of the nominal amounts bid at those prices. The PMP is expressed as a percentage of nominal value rounded up to 3 decimal places.

Bid assignment

- Assignment of bids is made as follows
  - bids at the marginal price are fulfilled at that price
  - those above the marginal price but below the PMP are fulfilled at the bid price
  - those at or above the PMP are fulfilled at the PMP

NOTE: At auctions where bids are denoted in terms of interest rates, bids with an interest rate that is less than or equal the weighted average interest rate will be assigned at the price which is equivalent to that average rate. Those with a higher interest rate will be assigned at the price corresponding to that higher interest rate

In both types of auctions, bid price and bid interest rate, all non-competitive bids will be fulfilled without exception, as long as at least one non-competitive bid is accepted. The price for the fulfillment of non-competitive bids is the PMP.

Auction example

There is an issue of Letras del Tesoro with a 12 month maturity (364 days later, non-leap year). Competitive bids are described on the following table. Non-competitive bids represent 150 million.

Precio Ofertado	Nominal Solicitado (millones euros)
95.250%	125
95.235	200
95.220	170
95.210	220
95,205	250
95.200	300

On the basis of this, el Tesoro decides to issue Letras for a nominal amount of 850 million euros.

The outcome

- All noncompetitive bids are accepted
- The remaining 700 million (850 − 150) is obtained from competitive bids, accepting those that offer the higher price and rejecting the others
- The marginal price (or minimum accepted price) was 95,210%, (those with lower prices were rejected)
- The following represents competitive bids and the PMP calculation

Precio Ofertado	Nominal Solicitado (millones euros)	Nominal Adjudicado Acumulado
95.250%	125	125
95.235	200	325
95.220	170	495
95.210	205	700

$$PMP = \frac{95.25 \times 125 + 95.235 \times 200 + 95.22 \times 170 + 95.21 \times 205}{700} = 95.227\%$$

Auction outcome

- Final prices

Precio Ofertado	Precio adjudicación	Nominal Solicitado (millones euros)
No compet	95.227	150
95.250%	95.227	125
95.235	95.227	200
95.220	95.22	170
95.210	95.21	205

- The amount of cash obtained from the issue is 809,32 m, on a nominal value of 850m which represents an average price of 95.214%

$$PM = 95.227\frac{475}{850} + 95.22\frac{170}{850} + 95.21\frac{205}{850} = 95.214$$

Prices and returns

- From these prices you can obtain the return obtained by each of these investors

Offered Price	Price Paid	Return
95.250%	95.227	4.957%
95.235%	95.227	4.957%
95.220%	95.220	4.964%
95.210%	95.210	4.975%

- These returns are obtained using the formula

$$95.227\,(1+r)^{\frac{364}{360}} = 100 \Rightarrow r = 5,17\%$$

Exercise
While you are reading the financial newspaper you come across the following information: el Tesoro sold Letras at the last auction with issue date 2-jan-2011, expiry 1-jan-2012. The return is 2.70%. Determine the price that investors who went to the auction paid for these Letras.

Example

- Consider that you bought assets via a financial intermediary at the PMP (95.227) but you are charged a commission of 5 per thousand on the issue, compute your real effective return.

- The interest on the investment was 5,17%

$$95.227\,(1+r)^{\frac{364}{360}} = 100 \Rightarrow r = 5,17\%$$

- But, taking into consideration the commission you pay, it is 4.44%

$$\begin{aligned} 95.227 + 95.227 \times 5/1000 &= 95.703 \\ 95.703\,(1+r)^{\frac{364}{365}} &= 100 \Rightarrow r = 4.633\% \end{aligned}$$

Class 13

REPOs

Market transactions

- In the money market you will find the following transactions using Letras

  Cash purchase/sale (al contado) Letras are bought and sold and paid for immediately in cash

  Delayed purchase/sale (a plazo) Letras are bought and sold but the Letras are not transferred until (at least) five days later

  REPOs (compraventa con pacto de recompra) There is an initial sale of a Letra, together with a commitment to buy the Letra back at a given (not so) future date at a given (fixed) price.

The repurchase price is set so that the initial buyer has a given guaranteed return. The most common is the REPO where the original owner continues to be so during the transaction. Thus, the "temporary sale" is organized so as to keep the right to collect coupon payments.

REPOs

REPO
Double transaction on a single asset and based on a single nominal value. The buyer does not have full ownership of the asset, and hence any transaction that uses this asset are denominated "in repo" until the asset returns to the original owner

- REPO characteristics
  - REPO is a definitive transaction: both parties are tied to the transaction until the resale (second simple operation) takes place
  - The buyer of the REPO makes a temporary acquisition (temporary investment) while the seller makes a temporary sale (temporary loan) of the asset
- A bank (A) is holding Letras and wants cash for one day. It decides to lend those Letras to another bank (B) who wants to invest some excess cash for one day.
- Bank A sells the Letras on Jan-2-2010 for 900.000€ and commits to repurchase them for 900.088,25€ the next day.
- Determine the annual return (using compound capn) obtained by Bank B

$$\begin{aligned} 900000\,(1+i)^{\frac{1}{365}} &= 900088,25 \\ i &= 0.0364 \Rightarrow i = 3.64\% \end{aligned}$$

Corporate Debt

The market for corporate bonds

- Non-financial firms also issue debt in secondary markets, although, the amount of debt issued is very small relative to that of government debt

Corporate bonds and commercial paper

- Commercial paper (pagaré) is a promise of payment made by large non financial corporations (Unión Fenosa, Renfe, Telefónica...).
- Commercial paper has long been a financial instrument but only recently has it become an openly traded asset in Spain, since 1982.

- These are negotiated on discount. The usual term is 1,3,6,12 and 18 meses. Nominal values vary between 1.500 and 6.000 euros. The secondary market where these are negotiated is the AIAF.
- Interest rates and prices are computed (as per circular de AIAF) using simple capitalization for terms less than 376 days and with a **365 day basis**. The law for effective returns is compound capitalization including all costs and commissions.

Example

- An investor buys commercial paper from firm XXX with 12 months to maturity and a nominal value of 1000 euros. Pays 980,02 euros with maturity in 364 days. Compute its return
  - As the return is less than 376 days we use simple capitalization

$$980.02\left(1+r\frac{364}{365}\right)=1000 \Rightarrow r=2.044\%$$

- The rule in AIAF is 365 days per year & if expiry $<$376 days use simple capitalization, otherwise use compounding

Example

- An investor bought commercial paper of a firm with nominal value 3000€ and maturity in 270 days and paid 2950.50€
  - Determine the official return on the investment
  - Suppose he wants to sell the paper at a future point in time, and he will obtain a price of 2.960€. Find the date at which he sells knowing that he obtains a yearly effective return of at least 2.25%

Solution

- The invesment is: invest $(2950.50, 0)$ to obtain $(3000, 270)$ in days. The return with simple capitalization

$$2950.50\left(1+r\frac{270}{365}\right)=3000 \Rightarrow r=2.27\%$$

- If you sell it today for 2.960 in t days, financial equivalence
  - with simple capitalization gives us

$$2950.50\left(1+0.0225\frac{t}{365}\right)=2960$$

    if $r=2.25\%$ then $t=52$ days
  - with compound capitalization

$$\begin{aligned} 2950.50\,(1+0.0225)^{\frac{t}{365}} &= 2960 \\ \frac{t}{365}\ln(1+0.0225) &= \ln\left(\frac{2960}{2950.50}\right) \end{aligned}$$

    you get $t=52.7$ (53) days

5 Installments

Theory

Class 15

Installments

- Concepts and classification of regular payments (installments)
- Constant rents (fixed installments)
- Differences between prepayable and postpayable installments

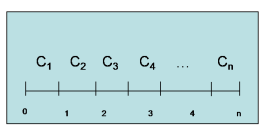
What is it?

- Quite certainly, in a few years you:
  - Will work in a firm and earn a monthly salary, e.g. 2000€ per month
  - Will buy a car and will probably finance it with monthly payments, e.g. 400€ per month
  - Will buy a house and pay a mortgage (e.g. 1000€ per month), or rent an apartment (maybe a bit less, 850€ per month)
  - Will invest in fixed income (e.g. bonds) and receive interest payments (coupons), for example once every six months (420€)

All these are examples of rents (payment by installments)

Concepts

A financial rent or a system of payment by installments is a sequence of successive payments, each with its own maturity



Components

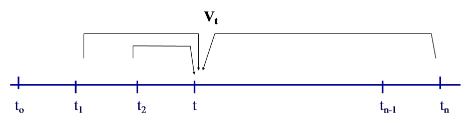
A financial rent has:

- Origin: lower end of the initial period
- End: upper end of last period
- Duration: time between origin and end
- Installment payment: each of the dated payments that make up the rent (total cash payment at each maturity date)

Our objective is to determine the financial value of these sequences of payments at any one particular point in time.

Financial value

- The (financial) value of a sequence of installments at a particular point in time is the financial sum of the value of each of the payments of the installments
- It can be calculated by financially transferring (capitalizing or discounting) each of the payments to a particular point in time (t)
- The valuation can be done at any point in time, at any t: at the origin of the rent, at the end of the rent, at an intermediate date, at a date before the origin, or at a date after the end of the rent



Value

Initial Value $[V_0]$ The value of the installment payments calculated at the origin of the rent

Final Value $[V_n]$ The value of the installment payments calculated at the end of the rent

- Theoretically, the valuation can be done using any of all the possible financial laws we have learnt, nevertheless ...

  We will be using the **Compounding** capitalization/discounting **law** with an effective interest rate r

Annuities and perpetuities

- Imagine that you want to determine the amount of money that you will need in order to be sure that you can pay the university fees for all 4 years of your degree. Suppose that the annual fee is 6.500€ and that the bank offers you a 10% effective APR on your deposits
- You can compute the present value of the upcoming payments
  - (6.500,1): VP $= 6500/(1 + 10\%) =$ 5.909,09€
  - (6.500,2): VP $= 6500/(1 + 10\%)^2 =$ 5.371,90€
  - (6.500,3): VP $= 6500/(1 + 10\%)^3 =$ 4.883,55€
  - (6.500,4): VP $= 6500/(1 + 10\%)^4 =$ 4.439,59€
  - Total: sum(VP) = 20.604,13€

Annuities and perpetuities

- From the above calculations, 20.604,13€ today is equivalent to 6.500€ per year for the next 4 years
- Equivalently, if you deposit 20.604,13€ in the bank today, and you receive a 10% effective APR on your deposits, you can stop worrying about university fees as you will be able to withdraw 6.500€ per year from the account at the end of each of the next four years and pay for the fees (plus, the balance at the end will be zero).

Annutities and perpetuities

- Note that we have no problem to determine the value of the rent with our current knowledge from previous Financial Mathematics classes
- In order to determine the present value of a sequence of future payments, at different dates, all we need to do is to discount each of those payments and sum the resulting amounts
- Discounting these payments individually, one by one, for a rent with 12, 52, or even 365 installments, can be very laborious, whether you are using a calculator, a worksheet, ... What we do is find a **shortcut**: a single formula that is able to summarize the value of all those future payments in a single calculation. There are several special cases where this is possible using relatively simple formulas

Annuities and perpetuities

- When faced with payment by installments and one wants to calculate the value of a sequence of such payments, one option is to discount all the payments and sum them
- Discounting each of the payments one by one can be very laborious/tedious. Note that the sum of the discounted values has a structure like the sum of a geometric series (which allows us to develop a useful shortcut/formula)
- Remember what a geometric series $a_1, a_2, \ldots, a_n$ looks like this:

$$a_1 = a_1, \quad a_2 = a_1 R, \quad a_3 = a_2 R = a_1 R^2, \quad a_n = a_1 R^{n-1}$$

- And we can remember (or derive) that the sum of these terms is equal to

$$S = \frac{a_1 - a_n R}{1 - R}$$

Annuities and perpetuities

- Where is the geometric series in the sum of discounted payments?
- Consider a series of payments of 1€ per year for n years, discounted at an effective APR of $r\%$
- Let $a_{n|r}$ denote the present value of the sum of this regular payments of 1€

$$a_{n|r} = PV = \frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^n}$$

- Let $R = 1/(1+r)$ and $a_1 = R$, then we have that

$$a_2 = \frac{1}{(1+r)^2} = R \cdot R = a_1 R, \quad a_n = \frac{1}{(1+r)^n} = a_1 R^{n-1}$$

- Then, from the formula for the sum of the terms of a geometric series:

$$S = \frac{a_1 - a_n R}{1 - R} = \frac{R - RR^n}{1 - R} = \frac{R(1 - R^n)}{1 - R} = \frac{(1 - R^n)}{\frac{1-R}{R}}$$

- where

$$\frac{1-R}{R} = \frac{1 - \frac{1}{1+r}}{\frac{1}{1+r}} = \frac{\frac{1+r}{1+r} - \frac{1}{1+r}}{\frac{1}{1+r}} = \frac{\frac{r}{1+r}}{\frac{1}{1+r}} = r$$

- so that

$$S = \frac{(1 - R^n)}{r} = \frac{\left(1 - \left(\frac{1}{1+r}\right)^n\right)}{r}$$

Annuity
The present value of an annuity is

$$a_{n|r} = \frac{1}{r}\left(1 - \left(\frac{1}{1+r}\right)^n\right)$$

Annuity

- For a sequence of constant payments of C€ each for n periods with a per period interest rate of r, the present value is

$$V_0 = PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \ldots + \frac{C}{(1+r)^n}$$

- If you take out C as a common factor you find that it is equal to the present value of a sequence of n payments of 1€ each times C

$$\begin{aligned} V_0 &= C\left(\frac{1}{1+r} + \frac{1}{(1+r)^2} + \ldots + \frac{1}{(1+r)^n}\right) \\ &= Ca_{n|r} \\ &= \frac{C}{r}\left(1 - \left(\frac{1}{1+r}\right)^n\right) \end{aligned}$$

Example

- You owe the last four payments on a personal loan of 4.000€ each. You are paying an interest rate of 10%. The firm you are working for has paid you a large sum of money as bonus for meeting your objectives, and you are considering whether to cancel the loan or not. Your bank says you can do it. How much money do you have to pay today to cancel the debt?

$$V_0 = 4000a_{n|r} = \frac{4000}{10\%}\left(1 - \left(\frac{1}{1+10\%}\right)^4\right) = 12,679.46€$$

The final value of a unitary rent

- In order to calculate the final value of a series of regular installment payments one can calculate the VF of each of the payments, capitalizing them to the end of the rent (using compounding)

- There is a related formula, $s_{n|r}$, which works like the one for the present value and can be derived in a similar way

$$s_{n|r} = 1\,(1+r)^{n-1} + 1\,(1+r)^{n-2} + \ldots + (1+r) + 1$$

- Just reorder the terms so that $a_1 = 1$, $a_2 = (1+r)$, ..., $a_n = (1+r)^{n-1}$ and use $R = (1+r)$, then

$$\begin{aligned} s_{n|r} &= S = \frac{a_1 - a_n R}{1-R} = \frac{1-(1+r)^{n-1}(1+r)}{1-(1+r)} \\ &= \frac{1-(1+r)^n}{-r} = -\frac{1-(1+r)^n}{r} \\ s_{n|r} &= \frac{(1+r)^n - 1}{r} \end{aligned}$$

and for a general rent of n payments of constant size C:

$$V_n = C s_{n|r}$$

Present and future value

- We have seen that the present value of a rent is: $V_0 = C a_{n|r}$ and its future value is $V_n = C s_{n|r}$
- These two are the values of the same sequence of payments with the same interest rate and financially law–hence they must be financially equivalent, i.e.

$$V_0 (1+r)^n = V_n \Rightarrow a_{n|r} (1+r)^n = s_{n|r}$$

Verify (at home) that this is indeed true

Example

- Your parents have decided to buy you a car when you graduate. In order to pay for it they will make four deposits of 6.500€ per year, at the end of each year, in an account which offers a 10% effective APR.
- How much money will there be in the account when you finish your fourth year?

$$\begin{aligned} V_0 &= 6.500 a_{4|10\%} = 20.604€ \\ V_n &= 6.500 s_{4|10\%} = 20.604 (1+10\%)^4 \\ &= 30.166,50€ \end{aligned}$$

Example

How much must I invest every year in order to have 100.000€ after 10 years if I have a deposit account which offers a 3% efective APR?

$$\begin{aligned} V_0 &= C a_{10|3\%} \\ V_{10} &= V_0 (1+3\%)^{10} = C a_{10|3\%} (1+3\%)^{10} \end{aligned}$$

As I want $V_{10} = 100.000$,

$$C = \frac{100.000}{C a_{10|3\%} (1+3\%)^{10}} = 8.723,05€$$

Classification of sequences of installments

By the amount of the installment	By duration	By relative frequency	By the first payment
Constant	Finite	Whole	Immediate
Variable	Infinite	Periodic	Advanced
		Fractional	Delayed

Classification

By the amount of the installment:

- Constant: all installment payments are of the same amount
- Variable: installment payments vary in their amounts

Classification

By duration:

- Finite: those that have a finite number of payments
- Infinite: those with an unlimited number of payments (also known as perpetuities)

Classification

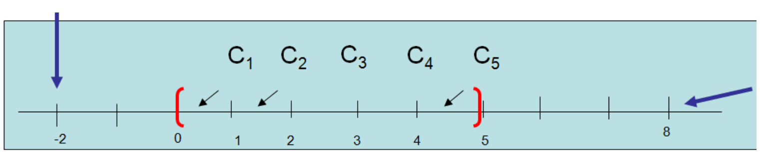
By the relative frequency:

- Whole: those for which the frequency of payments (e.g. annual) coincides with the capitalization period (e.g. annual interest payments)
- Periodic: those for which the frequency of payments (e.g. annual) is less frequent than that of the capitalization period (e.g. interest payments every month)
- Fractional: those for which the frequency of payments (e.g. monthly) is more frequent than that of the capitalization period (e.g. annual interest payments)

Classification

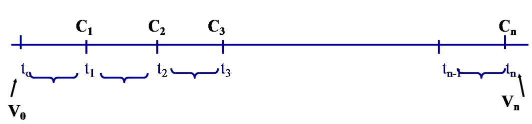
By the first payment

- Immediate: rent whose value is calculated at some point in between the origin and the end of the rent
- Delayed: rent whose value is calculated at some point before the origin of the rent
- Advanced: rent whose value is calculated at some point after the end of the rent

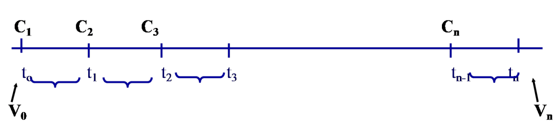


Prepayable and pospayable rents

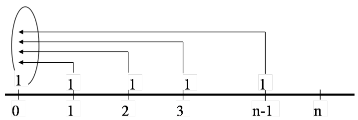
- So far we have been working with rents whose payment occurred at the end of the corresponding period. These are pospayable rents



- An alternative set of rents are those whose payment occurs at the beggining of the corresponding period–these are called prepayable rents



A prepayable, unitary, immediate, finite rent



Calculating the present value:

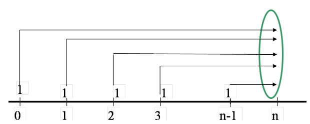
- discount all payments to date 0, i.e. the time of the first payment (**prepayable**)
- the formula is related to the (**pospayable**) annuity but to distinguish it we use two dots on top of the a:

$$\ddot{a}_{n|r} = a_{n|r}(1+r)$$

Prepayable annuity
The general formula for a constant prepayable annuity of C€ per period over n periods is

$$V_0 = C\ddot{a}_{n|r}$$

A prepayable, unitary, immediate, finite rent



For the final value we carry all payments to the end of the last period so that

$$\begin{aligned} V_0 &= C\ddot{s}_{n|r} \\ &= C\ddot{a}_{n|r}(1+r)^n \\ &= Ca_{n|r}(1+r)(1+r)^n \\ &= Ca_{n|r}(1+r)^{n+1} \end{aligned}$$

Example
You are renting a place for a business and you have to pay the rent every year, but in advance. The rent is 10.000€. You want to forget about the rent and you have a lot of liquidity now so you want to put money in the bank, which offers you a 6% efective APR, in order to take care of the rent for the next five years. How much do you have to put into the bank?

$$\begin{aligned} V_0 &= 10.000\ddot{a}_{5|6\%} = 10.000(1+6\%)a_{5|6\%} \\ &= 44.651,06€ \end{aligned}$$

Deferred and anticipated payments

Deferred payments: payments valued prior to the date of origin of the rent

Anticipated payments: payment valued later than the date of end of the rent

Deferred Payments

- Postpayable rent, deferred p periods:

$$V_{-p} = \frac{V_0}{(1+r)^p} = Ca_{n|r}(1+r)^{-p}$$

- Prepayable rent, deferred p periods:

$$V_{-p} = \frac{V_0}{(1+r)^p} = C\ddot{a}_{n|r}(1+r)^{-p} = Ca_{n|r}(1+r)^{1-p}$$

Anticipated payments

- Postpayable rent, anticipated p periods:

$$V_{n+p} = V_n (1+r)^p = C a_{n|r} (1+r)^{p+n}$$

- Prepayable rent, anticipated p periods:

$$V_{n+p} = V_n (1+r)^p = C \ddot{a}_{n|r} (1+r)^{n+p} = C a_{n|r} (1+r)^{1+n+p}$$

Example

You can pursue an investment and you will receive nothing for the first two years, but for the 3 years after that you will receive 6000€ per year. How much would you pay to make this investment if you minimum return you demand is 8%?

$$\begin{aligned} 6000 a_{3|8\%} &= V_2 = 6000 \frac{1-1.08^{-3}}{.08} = 15462.58 \\ V_0 &= V_2 (1.08)^{-2} = \frac{15462,58}{1.08^2} = 13,256.67€ \end{aligned}$$

Note that the rent can be evaluated as a prepayable rent but deferred for 3 years

Perpetual Payments

- Perpetual payments are sometimes used when there is no clear termination date for the rent (and hence, as an (over-)approximation)

$$a_{\infty|r} = \lim_{n\to\infty} a_{n|r} = \lim_{n\to\infty} \left(1-(1+r)^{-n}\right) \frac{1}{r} = \frac{1}{r}$$

- Generally, the formula for a perpetual rent of C€ per period is

$$C\frac{1}{r}$$

Prepayable perpetual rents

- If we consider a prepayable rent, and take the limit as n goes to infinity we obtain

$$\ddot{a}_{\infty|r} = (1+r)\, a_{\infty|r} = \frac{1+r}{r}$$

- In general, the value of a prepayable perpetual rent of C€ per period is

$$\ddot{V}_0 = C\frac{1+r}{r}$$

Example

Mr. Mendia works for a publishing house and is planning the possibility of setting up a poetry price with an annual award of 12000€. In order to finance this price, the publishing house will look for a donor. Juan Mendia is asked to compute how large the donation has to be taking into account that the publishing house can set up a foundation that will be able to deposit the money at an 8% annual rate and the price has indefinite duration. Can you help him?

$$\frac{12000}{8\%} = 150.000€$$

Variable payments

- We considered different kinds of sequences of payments, and made a difference between those with a constant term and those with a variable term
- For those with a fixed term, we have used a formula to simplify the computation of its value
- Similarly, there is a formula for sequences of variable terms, if those terms grow as a geometric progression, that is the payment at date t, C_t is related to that at $t-1$ by a contant growth term, g, i.e.

$$\begin{aligned} C_t &= (1+g)\, C_{t-1} \\ \Rightarrow C_t &= (1+g)^{t-1}\, C_1 \end{aligned}$$

- Note that these can also be prepayable, pospayable, differred, anticipated, ...

Rents in geometric progression

- An immediate, finite, postpayable rent in geometric progression
- The payouts are: $C_1 = C$, $C_2 = C_1 q$, $C_3 = C_2 q = Cq^2$, ..., $C_n = C_1 q^{n-1}$
- The constant term q describing the geometric progression can be
  - greater than 1, $q = 1+g$, $g > 0$: payments are growing over time, e.g. $g = 2\%$ $q = 1.02$
  - less than 1, $q = 1+g$, $g < 0$: payments are decreasing over time, e.g. $g = -2\%$ $q = 0.98$

Rents in geometric progression

- The formula used is

$$A\left(C,q\right)_{n|r} = C\left[\frac{1}{1+r} + \frac{1+g}{(1+r)^2} + \ldots + \frac{(1+g)^{n-1}}{(1+r)^n}\right]$$

- If
  - $q \neq 1+r$

$$A\left(C,q\right)_{n|r} = C\frac{1-(1+r)^{-n}\, q^n}{1+r-q} = C\frac{1-(1+r)^{-n}\,(1+g)^n}{r-g}$$

  - $q = 1+r$

$$A\left(C,q\right)_{n|r} = \frac{nC}{1+r}$$

Example

Suppose you have to calculate the present value of future annual salaries of one million euros for the next five years, and which grow at a rate of 2%, where the discount rate is 3% per year.

$$V_0 = A\left(1m, 1.02\right)_{5|3\%} = 1m \times \frac{1-(1.02/1.03)^5}{3\% - 2\%} = 4.76m$$

Perpetual rents in geometric progression

- Some of these rents do not have a natural ending date and it is convenient to consider that it grows for ever

$$\begin{aligned} A(C,q)_{\infty|r} &= \lim_{n\to\infty} A(C,q)_{n|r} = \lim_{n\to\infty} C\frac{1-q^n(1+r)^{-n}}{1+r-q} \\ &= \lim_{n\to\infty} C\frac{1}{1+r-q}\left(1-\left(\frac{q}{1+r}\right)^n\right) \end{aligned}$$

- If $q < 1+r$ then $\lim_{n\to\infty}(q/(1+r))^n = 0$ and the formula is

$$A(C,q)_{\infty|r} = \frac{C}{1+r-q}$$

- If $q > 1+r$ then $\lim_{n\to\infty}(q/(1+r))^n = \infty$
- If $q = 1$ then $\lim_{n\to\infty} A(C,q)_{n|r} = \lim_{n\to\infty} nC(1+r) = \infty$
- In both of the previous last two cases the value is infinite

Example
Juan Mendía is not very convinced about the idea of the literary price. He does not think it reasonable that the 12000 euros annual price should be constant, because it would mean that the relative value of the price will be lower each year. In 20 years its value will be very low. Hence, he wants to calculate how big the donation has to be in order for the foundation to have a 12000 euros price in the first year, but thereafter have it increase at a 3% per year. Remember, the interest rate is 8% and do the calculation so that the money is able to cover payments for the next 20 years. [As an exercise, do it also so that the donation will be able to cover the prices forever]

Example
You have bought an apartment whose cash value is 150.000 € but you agree with the seller to make the payments in the following way: 30.000 € at the time of signing the contract; the rest is payable annually in five payments with the same nominal value. You have two options to chose from:

1. the first payment due next year, or
2. the first payment due six years after the contract is signed

The interest rate is 5% efective APR. Compute the payments in the two possible alternative schemes.

Example
James signs up with a financial institution to set up a savings plan. He wants to save one million euros over the next ten years. Compute how much he has to save at the beginning of each year if the institution offers an annual effective APR of 3% and deposits will be made only during the first five years.

Example
A firm has signed a maintenance contract. The payments are 6.000 euros for the first year and an annual increase of 8% per year. If you use an interest rate of 12% per year, compute the present value of this payment stream if they are prepayable and of indefinite duration.

Periodic Installments

- Installment payments are usually regularly paid out and thus naturally periodic
- But, we also use the term "periodic installments" to refer to installment payments whose frequency is smaller than that of interest capitalization
- Thus, they refer to payments who happen infrequently (once every two years) relative to the frequency of interest payments, which are more frequent (yearly)

Periodic Installments
Periodic installments are evaluated as regular rents, and the only adjustment required is to recalculate the period of the interest rate to match that of the payments.

Example
Suppose you run a transportation firm which requires tire changes for all the trucks once every two years. The cost is 20.000€ every two years. Compute the expected cost of the tires for the next 10 years using an effective APR of 5%

- We first change the interest rate to match the periodicity of the payments

$$\begin{aligned} (1+r) &= (1+5\%)^2 = 1.1025 \\ \Rightarrow \quad r &= 10.25\% \end{aligned}$$

- Then we compute the value of the payments

$$V_0 = 20000 a_{5|10.25\%} = 75334€$$

Fractional Installments

- Fractional installments are characterized by payments that are more frequent than the capitalization of interest
- Thus, fractional installments refer to payments who happen frequently (every month) relative to the frequency of interest payments, which are less frequent (yearly)

Fractional Installments
Fractional installments are evaluated as regular rents, and the only adjustment required is to recalculate the period of the interest rate to match that of the payments.

Method

- Let m be the number of payments made during the period of capitalization
- Example, monthly payments with annual (**effective**) interest rate. Then, $m = 12$

$$\begin{aligned} \left(1+r^{(m)}\right)^m &= 1+r \\ r^{(m)} &= (1+r)^{1/m} - 1 \end{aligned}$$

- Then, use $Ca_{n|r^{(m)}}$

Example: monthly payments

- An employee of a multinational corporation receives a monthly salary of 2000€ (after tax) at the end of each month
- Fortunately, he lives with his family so that he only needs to cover his expenses and has been able to save 1200€ every month
- Compute the amount he has saved for 36 months in an account that gives him a 3% nominal APR with monthly capitalization

Example

- The first thing to note is that the interest rate given is NOT effective. Usually, banks give an annual interest rate which, when not explicitly stated as effective, is a **nominal** interest rate
- Hence, the rent is not really a fractional rent–the capitalization period is the same as that of the account. It is a regular monthly installment calculation, with the caveat that we have to transform the quoted rate into the effective monthly rent

$$r = \frac{3\%}{12} = 0.25\%$$

- Then, the final value can be computed in the usual way

$$\begin{aligned} V_{36} &= V_0 (1+r)^{36} \\ &= 1200 a_{36|.25\%} (1+0.25\%)^{36} \\ &= 1200 \times 34.39 (1+0.25\%)^{36} = 45144.67 \end{aligned}$$

Perpetual rent example

- Your firm wants to offer a price of a monthly salary of 1.000 euros, which, in case of death, will be inherited by his or her descendants. Compute the quantity the firm has to set aside in order to be able to guarantee this obligation–payments start next year (at the end of the month). The annual interest rate is 3% on the savings account with monthly compounding
- The interest rate is

$$r^{(12)} = \frac{3\%}{12} = 0.25\%$$

- The value of the payments is

$$V = \frac{1000}{.25\%} = 400000$$

- And it starts one year from today

$$V_0 = \frac{400000}{(1+.25\%)^{12}} = 388192.75$$

Example

Sara has just started her studies, which she plans of prolonging over five years. Sara's grandfather promised her that he would deposit 4800 euros each year in a bank account at a 12% effective APR, and give her the balance at the end. He is going to make these deposits at the end of each quarter (1200€ per quarter). How much will she be getting?

Example

- A colleague of your mother's, Joan, has decided to sign up for a pension plan, so that he can have the money when he retires. From the information provided by the bank, you gather the interest rate used is 5% nominal APR with monthly compounding.
- Joan is going to deposit a fixed amount on the first of each month, namely 300€, every month. After three years (and before making the deposit for the new month), he changes his mind–he feels too young to have a pension plan–so he decides to spend the money
- The deposit institution charges a 1% penalty on the remaining balance for early withdrawal. Determine
  1. The amount of money he has to spend
  2. The real effective annual return he has obtained from his savings

Life insurance

- Whole life insurance is an increasingly popular way to save in order to have additional income upon retirement.
- The insurance company ensures the investor a lifetime income (that could be monthly, quarterly, ...) until his death, in exchange for an initial, single lumpsum payment. The contract includes an attractive implicit interest rate
- In addition to the "salary", the insurance company offers a death benefit, that is, in case of death, the insurance company pays the insured's beneficiaries
- There are several versions
  - Immediate: the insured receives the payments from the moment the contract is signed.
  - Deferred: the insured starts to receive payments after a fixed future date.

Example

- A woman, aged 50, has just inherited 300.000€ and wants to invest all of it in whole life insurance.
- She goes to her insurance broker, who (using the expected lifetime of a woman in her situation, 83 years) determines the monthly amount she will receive from now until her (expected) death. The interest guarantee is 3% of the nominal amount.
- At the woman's death, her inheritors will receive the initial premium amount.
- Determine the woman's monthly "income" from this contract