# Chapter 2

# Limits and continuity of functions of one variable

## 2.1 Limits

To determine the behavior of a function $f$ as $x$ approaches a finite value $c$, we use the concept of limit. We say that the limit of $f$ is $L$, and write $\lim_{x\to c} f(x) = L$, if the values of $f$ approaches $L$ when $x$ gets closer to $c$.

**Definición 2.1.1.** (Limit when $x$ approach a finite value $c$). We say that $\lim_{x\to c} f(x) = L$ if for any small positive $\epsilon$, there is a positive $\delta$ such that

$$|f(x) - L| < \epsilon$$

whenever $0 < |x - c| < \delta$.

We can split the above definition in two parts, using one–sided limits.

**Definición 2.1.2.**

1. We say that $L$ is the limit of $f$ as $x$ approaches $c$ from the right, $\lim_{x\to c^+} f(x) = L$, if for any small positive $\epsilon$, there is a positive $\delta$ such that

   $$|f(x) - L| < \epsilon$$

   whenever $0 < x - c < \delta$.

2. We say that $L$ is the limit of $f$ as $x$ approaches $c$ from the left, $\lim_{x\to c^-} f(x) = L$, if for any small positive $\epsilon$, there is a positive $\delta$ such that

   $$|f(x) - L| < \epsilon$$

   whenever $0 < c - x < \delta$.

**Teorema 2.1.3.** *$\lim_{x\to c} f(x) = L$ if and only if*

$$\lim_{x\to c^+} f(x) = L \quad \textit{and} \quad \lim_{x\to c^-} f(x) = L.$$

We can also wonder about the behavior of the function $f$ when $x$ approaches $+\infty$ or $-\infty$.

**Definición 2.1.4.** (Limits when $x$ approaches $\pm\infty$)

1. $\lim_{x\to+\infty} f(x) = L$ if for any small positive $\epsilon$, there is a positive value of $x$, call it $x_1$, such that
$$|f(x) - L| < \epsilon$$
whenever $x > x_1$.

2. $\lim_{x\to-\infty} f(x) = L$ if for any small positive $\epsilon$, there is a negative value of $x$, call it $x_1$, such that
$$|f(x) - L| < \epsilon$$
whenever $x < x_1$.

If the absolute values of a function become arbitrarily large as $x$ approaches either a finite value $c$ or $\pm\infty$, then the function has no finite limit $L$ but will approach $-\infty$ or $+\infty$. It is possible to give the formal definitions. For example, we will say that $\lim_{x\to c} f(x) = +\infty$ if for any large positive number $M$, there is a positive $\delta$ such that

$$f(x) > M$$

whenever $0 < |x - c| < \delta$. Please, complete the remaining cases.

**Nota 2.1.5.** Note that it could be $c \in D(f)$, so $f(c)$ is well defined, but $\lim_{x\to c} f(x)$ does not exits or $\lim_{x\to c} f(x) \neq f(c)$. Consider for instance the function $f$ that is equal to 1 for $x \neq 0$, but $f(0) = 0$. Then clearly the limit of $f$ at 0 is $1 \neq f(0)$.

**Ejemplo 2.1.6.** Consider the following limits.

1. $\lim\limits_{x\to 6} x^2 - 2x + 7 = 31$.

2. $\lim\limits_{x\to\pm\infty} x^2 - 2x + 7 = \infty$, because the leading term in the polynomial gets arbitrarily large.

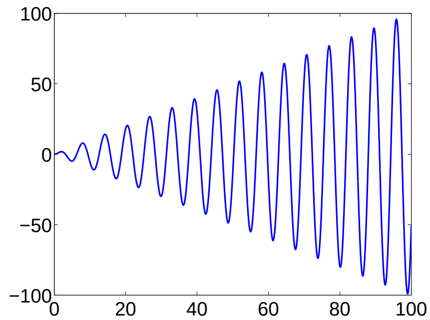
3. $\lim\limits_{x\to+\infty} x^3 - x^2 = \infty$, because the leading term in the polynomial gets arbitrarily large for large values of $x$, but $\lim\limits_{x\to-\infty} x^3 - x^2 = -\infty$ because the leading term in the polynomial gets arbitrarily large in absolute value, and negative.

4. $\lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$, since for $x$ arbitrarily large in absolute value, $1/x$ is arbitrarily small.

5. $\lim\limits_{x\to 0} \dfrac{1}{x}$ does not exists. Actually, the one–sided limits are:
$$\lim_{x\to 0^+} \frac{1}{x} = +\infty.$$
$$\lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

The right limit is infinity because $1/x$ becomes arbitrarily large when $x$ is small and positive. The left limit is minus infinity because $1/x$ becomes arbitrarily large in absolute value and negative, when $x$ is small and negative.

6. $\lim_{x\to+\infty} x \operatorname{sen} x$ does not exist. As $x$ approaches infinity, $\operatorname{sen} x$ oscillates between 1 and $-1$. This means that $x \operatorname{sen} x$ changes sign infinitely often when $x$ approaches infinity, whilst taking arbitrarily large absolute values. The graph is shown below.




7. Consider the function $f(x) = \begin{cases} x^2, & \text{if } x \le 0; \\ -x^2, & \text{if } 0 < x \le 1; \\ x, & \text{if } x > 1. \end{cases}$ $\lim_{x\to 0} f(x) = f(0) = 0$, but $\lim_{x\to 1} f(x)$ does not exist since the one–sided limits are different.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x = 1,$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} -x^2 = -1.$$

8. $\lim_{x\to 0} \frac{|x|}{x}$ does not exist, because the one–sided limits are different.

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = 1,$$
$$\lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = -1 \qquad (\text{when } x \text{ is negative, } |x| = -x).$$

In the following, $\lim f(x)$ refer to the limit as $x$ approaches $+\infty$, $-\infty$ or a real number $c$, but we never mix different type of limits.

### 2.1.1 Properties of limits

$f$ and $g$ are given functions and we suppose that all the limits below exist; $\lambda \in \mathbb{R}$ denotes an arbitrary scalar.

1. *Product by a scalar:* $\lim \lambda f(x) = \lambda \lim f(x)$.
2. *Sum:* $\lim(f(x) + g(x)) = \lim f(x) + \lim g(x)$.

3. *Product:* $\lim f(x)g(x) = (\lim f(x))(\lim g(x))$.

4. *Quotient:* If $\lim g(x) \neq 0$, then $\lim \dfrac{f(x)}{g(x)} = \dfrac{\lim f(x)}{\lim g(x)}$.

**Teorema 2.1.7** (Squeeze Theorem)**.** *Assume that the functions $f$, $g$ and $h$ are defined around the point $c$, except, maybe, for the point $c$ itself, and satisfy the inequalities*

$$g(x) \leq f(x) \leq h(x).$$

*Let* $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$. *Then*

$$\lim_{x\to c} f(x) = L.$$

**Ejemplo 2.1.8.** Show that $\displaystyle\lim_{x\to 0} x \operatorname{sen}\left(\frac{1}{x}\right) = 0$.

SOLUTION: We use the theorem above with $g(x) = -|x|$ and $h(x) = |x|$. Notice that for every $x \neq 0$, $-1 \leq \operatorname{sen}(1/x) \leq 1$ thus, when $x > 0$

$$-x \leq x \operatorname{sen}(1/x) \leq x,$$

and when $x < 0$

$$x \leq x \operatorname{sen}(1/x) \leq -x.$$

These inequalities mean that $-|x| \leq x \operatorname{sen}(1/x) \leq |x|$. Since

$$\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0,$$

we can use the theorem above to conclude that $\lim_{x\to 0} x \operatorname{sen} \frac{1}{x} = 0$.

### 2.1.2 Techniques for evaluating $\lim \dfrac{f(x)}{g(x)}$

1. Use the property of the quotient of limits, if possible.

2. If $\lim f(x) = 0$ and $\lim g(x) = 0$, try the following:

   (a) Factor $f(x)$ and $g(x)$ and reduce $\frac{f(x)}{g(x)}$ to lowest terms.

   (b) If $f(x)$ or $g(x)$ involves a square root, then multiply both $f(x)$ and $g(x)$ by the conjugate of the square root.

   **Ejemplo 2.1.9.**

$$\lim_{x\to 3} \frac{x^2-9}{x+3} = \lim_{x\to 3} \frac{(x-3)(x+3)}{x+3} = \lim_{x\to 3}(x-3) = 0.$$

$$\lim_{x\to 0} \frac{1-\sqrt{1+x}}{x} = \lim_{x\to 0} \frac{1-\sqrt{1+x}}{x}\left(\frac{1+\sqrt{1+x}}{1+\sqrt{1+x}}\right) = \lim_{x\to 0} \frac{-x}{x(1+\sqrt{1+x})} = \lim_{x\to 0} \frac{-1}{1+\sqrt{1+x}} = -\frac{1}{2}.$$

3. If $f(x) \neq 0$ and $\lim g(x) = 0$, then either $\lim \frac{f(x)}{g(x)}$ does not exist or $\lim \frac{f(x)}{g(x)} = +\infty$ or $-\infty$.

4. If $x$ approaches $+\infty$ or $-\infty$, divide the numerator and denominator by the highest power of $x$ in any term of the denominator.

   **Ejemplo 2.1.10.**

$$\lim_{x\to\infty} \frac{x^3 - 2x}{-x^4 + 2} = \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{2}{x^3}}{-1 + \frac{2}{x^4}} = \frac{0-0}{-1+0} = 0.$$

### 2.1.3 Exponential limits

Let the limit

$$\lim_{x\to c} [f(x)]^{g(x)}$$

be an indetermination. This happens if

- $\lim_{x\to c} f(x) = 1$ and $\lim_{x\to c} g(x) = \infty$ $(1^\infty)$.
- $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$ $(0^0)$.
- $\lim_{x\to c} f(x) = \infty$ and $\lim_{x\to c} g(x) = 0$ $(\infty^0)$.

Noting that

$$\lim_{x\to c} [f(x)]^{g(x)} = \lim_{x\to c} e^{g(x)\ln f(x)} = e^{\lim_{x\to c} g(x)\ln f(x)},$$

all cases are reduced to the indetermination $0 \cdot \infty$, since we have to compute the limit

$$\lim_{x\to c} g(x)\ln f(x).$$

In the first indetermination, $1^\infty$, it often helps to use the identity

$$\lim_{x\to c} g(x)\ln f(x) = \lim_{x\to c} g(x)(f(x) - 1).$$

since when $x$ is close to 0, $\ln(1+x) \approx x$, or, $\ln x \approx x - 1$

**Ejemplo 2.1.11.** $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1+\frac{1}{x}\right)} = e^{x\frac{1}{x}} = e.$

**Ejemplo 2.1.12.** Let $a, b > 0$. Calculate $\lim_{x\to\infty} \left(\frac{1+ax}{2+bx}\right)^x$.

If $a > b$, then the basis function tends to $a/b > 1$, thus the limit is $\infty$. If $a < b$, then the basis function tends to $a/b < 1$, thus the limit is 0. When $a = b$

$$\lim_{x\to\infty} \left(\frac{1+ax}{2+ax}\right)^x = e^{\lim_{x\to\infty} x\left(\frac{1+ax}{2+ax} - 1\right)} = e^{\lim_{x\to\infty} \frac{-x}{2+ax}} = e^{-1/a}.$$

### 2.1.4 Remarkable limit

Recall that

$$\lim_{x\to 0} \frac{\operatorname{sen} x}{x} = 1.$$

**Ejemplo 2.1.13.** Evaluate the following limits:

1. $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \frac{1}{\cos x} = \lim_{x\to 0} \frac{\operatorname{sen} x}{x} \lim_{x\to 0} \frac{1}{\cos x} = 1 \cdot 1 = 1.$

2. $\lim_{x\to 0}\frac{\operatorname{sen} 3x}{x} \overset{\{z=3x\}}{=} \lim_{z\to 0}\frac{\operatorname{sen} z}{\frac{z}{3}} = 3\lim_{z\to 0}\frac{\operatorname{sen} z}{z} = 3.$

## 2.2 Asymptotes

An *asymptote* is a line that the graph of a function approaches more and more closely until the distance between the curve and the line almost vanishes.

**Definición 2.2.1.** Let $f$ be a function

1. The line $x = c$ is a vertical asymptote of $f$ if $\lim_{x\to c^+}|f(x)| = \infty$ or $\lim_{x\to c^-}|f(x)| = \infty$.
2. The line $y = b$ is a horizontal asymptote of $f$ if $\lim_{x\to +\infty} f(x) = b$ or $\lim_{x\to -\infty} f(x) = b$.
3. The line $y = ax + b$ is an oblique asymptote of $f$ if
   (a) $\lim\limits_{x\to +\infty}\frac{f(x)}{x} = a$ and $\lim\limits_{x\to +\infty}(f(x) - ax) = b$, or
   (b) $\lim\limits_{x\to -\infty}\frac{f(x)}{x} = a$ and $\lim\limits_{x\to -\infty}(f(x) - ax) = b$.

Notice that a horizontal asymptote is a particular case of oblique asymptote with $a = 0$.

**Ejemplo 2.2.2.** Determine the asymptotes of $f(x) = \dfrac{(1+x)^4}{(1-x)^4}$.

Solution: Since the denominator vanishes at $x = 1$, the domain of $f$ is $\mathbb{R} - \{1\}$. Let us check that $x = 1$ is a vertical asymptote of $f$:

$$\lim_{x\to 1^\pm}\frac{(1+x)^4}{(1-x)^4} = +\infty$$

On the other hand

$$\lim_{x\to +\infty}\frac{(1+x)^4}{(1-x)^4} = \lim_{x\to +\infty}\frac{(1/x+1)^4}{(1/x-1)^4} = 1$$

hence $y = 1$ is a horizontal asymptote at $+\infty$. In the same way, $y = 1$ is a horizontal asymptote at $-\infty$. There is no other oblique asymptotes.

**Ejemplo 2.2.3.** Determine the asymptotes of $f(x) = \dfrac{3x^3 - 2}{x^2}$.

Solution: The domain of $f$ is $\mathbb{R} - \{0\}$. Let us check that $x = 0$ is a vertical asymptote of $f$.

$$\lim_{x\to 0^\pm}\frac{3x^3-2}{x^2} = \lim_{x\to 0^\pm}(3x - \frac{2}{x^2}) = \lim_{x\to 0^\pm} 3x - \lim_{x\to 0^\pm}\frac{2}{x^2} = -\infty.$$

Thus, $x = 0$ is a vertical asymptote of $f$. On the other hand

$$\lim_{x\to \pm\infty}\frac{3x^3-2}{x^2} = \lim_{x\to \pm\infty}(3x - \frac{2}{x^2}) = \pm\infty$$

thus, there is no horizontal asymptote. Let us study now oblique asymptotes:

$$a = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{3x^3-2}{x^3} = \lim_{x\to\pm\infty} \left(3 - \frac{2}{x^3}\right) = 3,$$
$$b = \lim_{x\to\pm\infty} (f(x) - 3x) = \lim_{x\to\pm\infty} \left(\frac{3x^3-2}{x^2} - 3x\right) = \lim_{x\to\pm\infty} \left(-\frac{2}{x^2}\right) = 0.$$

We conclude that $y = 3x$ is an oblique asymptote both at $+\infty$ and $-\infty$.

## 2.3 Continuity

The easiest limits to evaluate are those involving continuous functions. Intuitively, a function is continuous if one can draw its graph without lifting the pencil from the paper.

**Definición 2.3.1.** A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $c$ if $c \in D(f)$ and

$$\lim_{x\to c} f(x) = f(c).$$

Hence, $f$ is *discontinuous at* $c$ if either $f(c)$ is undefined or $\lim_{x\to c} f(x)$ does not exist or $\lim_{x\to c} f(x) \neq f(c)$.

### 2.3.1 Properties of continuous functions

Suppose that the functions $f$ and $g$ are both continuous at $c$. Then the following functions are also continuous at $c$.

1. *Sum.* $f + g$.
2. *Product by a scalar.* $\lambda f$, $\lambda \in \mathbb{R}$.
3. *Product.* $fg$.
4. *Quotient.* $f/g$, whenever $g(c) \neq 0$.

### 2.3.2 Continuity of a composite function

Suppose that $f$ is continuous at $c$ and $g$ is continuous at $f(c)$. Then, the composite function $g \circ f$ is also continuous at $c$.

### 2.3.3 Continuity of elementary functions

A function is called *elementary* if it can be obtained by means of a finite number of arithmetic operations and superpositions involving basic elementary functions. The functions $y = C =$ constant, $y = x^a$, $y = a^x$, $y = \ln x$, $y = e^x$, $y = \operatorname{sen} x$, $y = \cos x$, $y = \tan x$, $y = \arctan x$ are examples of elementary functions. *Elementary functions are continuous in their domain.*

**Ejemplo 2.3.2.**

1. The function $f(x) = \sqrt{4-x^2}$ is the composition of the functions $y = 4 - x^2$ and $f(y) = y^{1/2}$, which are elementary, thus $f$ is continuous in its domain, that is, in $D = [-2, +2]$.

2. The function $g(x) = \frac{1}{\sqrt{4-x^2}}$ is the composition of function $f$ above and function $g(y) = 1/y$, thus it is elementary and continuous in its domain, $D(g) = (-2, +2)$.

### 2.3.4 Limit of a composite function

Let $f, g$ be functions from $\mathbb{R}$ to $\mathbb{R}$ and $c \in \mathbb{R}$. If $g$ is continuous at $L$ and $\lim_{x\to c} f(x) = L$, then

$$\lim_{x\to c} g(f(x)) = g(\lim_{x\to c} f(x)) = g(L).$$

**Ejemplo 2.3.3.** Show that $\lim\limits_{x\to 1} \arctan\left(\dfrac{x^2+x-2}{3x^2-3x}\right) = \dfrac{\pi}{4}$.

SOLUTION: The function $\tan^{-1}$ is continuous.

$$\begin{aligned}
\lim_{x\to 1} \arctan\left(\frac{x^2+x-2}{3x^2-3x}\right) &= \arctan\left(\lim_{x\to 1} \frac{x^2+x-2}{3x^2-3x}\right)\\
&= \arctan\left(\lim_{x\to 1} \frac{(x-1)(x+2)}{3x(x-1)}\right)\\
&= \arctan\left(\lim_{x\to 1} \frac{x+2}{3x}\right)\\
&= \arctan 1\\
&= \frac{\pi}{4}.
\end{aligned}$$

**Ejemplo 2.3.4.** Evaluate the following limits:

- $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = \lim\limits_{x\to 0} \ln(1+x)^{1/x} = \ln\left(\lim\limits_{x\to 0}(1+x)^{1/x}\right) = \ln e = 1.$

  Notice that the function $\ln(\cdot)$ is continuous at $e$, thus we can apply 2.3.4.

- $\lim\limits_{x\to 0} \dfrac{a^x-1}{x} = \lim\limits_{z\to 0} \dfrac{z}{\frac{\ln(1+z)}{\ln a}} = \ln a \left(\lim\limits_{z\to 0} \dfrac{z}{\ln(1+z)}\right) = \ln a.$

  We have used $z = a^x - 1$, so that $x = \ln(1+z)/\ln a$, and the value of the limit above.

### 2.3.5 Continuity theorems

Continuous functions have interesting properties. We shall say that a function is continuous in the *closed* interval $[a, b]$ if it is continuous at every point $x \in [a, b]$.

**Teorema 2.3.5** (Bolzano's Theorem)**.** *If $f$ is continuous in $[a, b]$ and $f(a)\cdot f(b) < 0$, then there exists some $c \in (a, b)$ such that $f(c) = 0$.*

**Ejemplo 2.3.6.** Show that the equation $x^3 + x - 1 = 0$ admits a solution, and find it with an error less than 0.1.

SOLUTION: With $f(x) = x^3 + x - 1$ the problem is to show that there exists $c$ such that $f(c) = 0$. We want to apply Bolzano's Theorem. First, $f$ is continuous in $\mathbb{R}$. Second, we identify a suitable interval $I = [a, b]$. Notice that $f(0) = -1 < 0$ and $f(1) = 1 > 0$ thus, there is a solution $c \in (0, 1)$.

Now, to find an approximate value for $c$, we use a method of *interval–halving* as follows: consider the interval $[0.5, 1]$; $f(0.5) = 1/8 + 1/2 - 1 < 0$ and $f(1) > 0$, thus $c \in (0.5, 1)$. Choose now the interval $[0.5, 0.75]$; $f(0.5) < 0$ and $f(0.75) = 27/64 + 3/4 - 1 > 0$ thus, $c \in (0.5, 0.75)$. Let now the interval $[0.625, 0.75]$; $f(0.625) \approx -0.13$ and $f(0.74) > 0$ thus, $c \in (0.625, 0.75)$. The solution is approximately $c = 0.6875$ with a maximum error of $0.0625$.

**Teorema 2.3.7** (Weierstrass' Theorem)**.** *If $f$ is continuous in $[a, b]$ , then there exist points $c, d \in [a, b]$ such that*

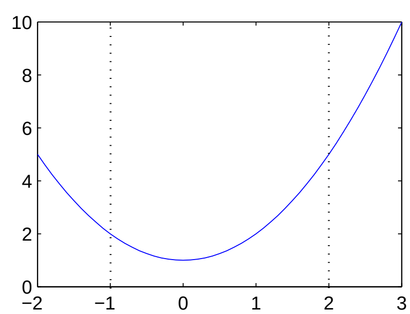
$$f(c) \leq f(x) \leq f(d)$$

*for every $x \in [a, b]$.*

The theorem asserts that a continuous function attains over a closed interval a minimum ($m = f(c)$) and a maximum value ($M = f(d)$). The point $c$ is called a *global minimizer* of $f$ on $[a, b]$ and $d$ is called a *global maximizer* of $f$ on $[a, b]$.

**Ejemplo 2.3.8.** Show that the function $f(x) = x^2 + 1$ attains over the closed interval $[-1, 2]$ a minimum and a maximum value.

SOLUTION: The graph of $f$ is shown below.




We can see that $f$ is continuous in $[-1, 2]$, actually $f$ is continuous in $\mathbb{R}$, and $f$ attains the minimum value at $x = 0$, $f(0) = 1$, and the maximum value at $x = 2$, $f(2) = 5$.

**Ejemplo 2.3.9.** The assumptions in the Theorem of Weierstrass are essential.

- The interval is not closed, or not bounded.
  - Take $I = (0, 1]$ and $f(x) = 1/x$; $f$ is continuous in $I$, but it does not have global maximum.

  - Take $I = [0, \infty)$ and $f(x) = 1/(1+x)$; $f$ is continuous in $I$, but it does not have global minimum, since $\lim_{x \to \infty} f(x) = 0$, but $f(x) > 0$ is strictly positive for every $x \in I$.

- The function is not continuous. Take $I = [0, 1]$ and $f(x) = \begin{cases} x, & \text{if } 0 \leq x < 1; \\ 0, & \text{if } x = 1. \end{cases}$ ; $f$ has a global minimum at $x = 0$, but there is no global maximum since $\lim_{x \to 1} f(x) = 1$ but $f(x) < 1$ for every $x \in I$.