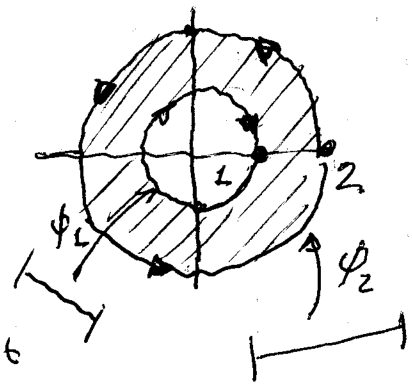
AM Prácticas 18-12-2020

8. $U = \{1 \leq x^2+y^2 \leq 4\}$; $f(x,y) = \begin{cases} 1 & \text{si } x^2+y^2=4 \\ x^2 & \text{si } x^2+y^2=1 \end{cases}$

Calcula $\int_U d\omega$, siendo $\omega = -yf\,dx + xf\,dy$

S/ Stokes

$$\int_U d\omega = \int_{\partial U} \omega = \int_{\phi_1} \omega + \int_{\phi_2} \omega$$



$\phi_2(t) = (2\cos t, 2\operatorname{sen} t)$, $0 \leq t \leq 2\pi$

$$\int_{\phi_2} \omega = \int_0^{2\pi} \phi_2^* \omega = \int_0^{2\pi} +(2\operatorname{sen} t)^2 1\,dt + (2\cos t)^2 1\,dt$$

$$= \int_0^{2\pi} 4\,dt = 8\pi$$

$\phi_1(t) = (\cos t, -\operatorname{sen} t)$, $0 \leq t \leq 2\pi$

$$\int_{\phi_1} \omega = \int_0^{2\pi} \phi_1^* \omega = \int_0^{2\pi} -(\operatorname{sen} t)^2 (\cos^2 t)\,dt - (\cos t)^4\,dt$$

$$= \int_0^{2\pi} -\cos^2 t \left[\operatorname{sen}^2 t + \cos^2 t\right] dt = -\int_0^{2\pi} \cos^2 t\,dt =$$

$$= \int_0^{2\pi} -\frac{1+\cos 2t}{2}\,dt = -(\pi + 0) = -\pi$$

$$\int_U d\omega = 8\pi - \pi = 7\pi$$

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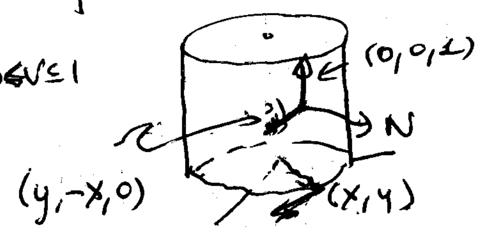
9. $C = \{x^2+y^2=1,\ 0<z<1\}$

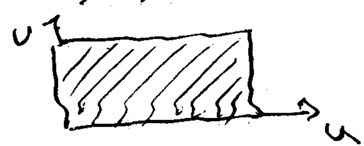
$\omega = (\operatorname{sen}(\pi z)\, e^{x^2+y^4} + y e^{z})\,dx + z^2\,dy + e^{x+y}\,dz$

a) $N(x,y,z)=(x,y,0)$ normal unitaria para C



$\phi(u,v) = (\cos u, \operatorname{sen} u, v)$, $0\le u\le 2\pi$, $0\le v\le 1$

$\phi_u = (-\operatorname{sen} u, \cos u, 0)$, $\phi_v=(0,0,1)$



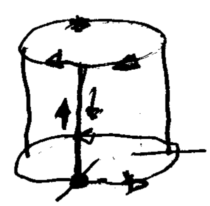
$\phi_u\times\phi_v = (\cos u, \operatorname{sen} u, 0) = (x,y,0) = N$

$\|N\| = \sqrt{x^2+y^2} = 1$

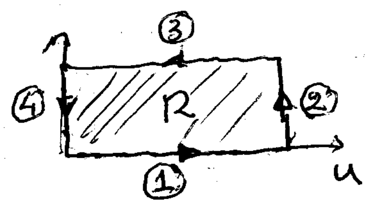
Es claro que $u=(0,0,1)$ es tangente a C y tambien $v=(y,-x,0)$ es tangente a C en (x,y,z). Entonces

$\langle u, N\rangle = 0$ y $\langle v, N\rangle = 0 \Rightarrow N$ perp a C

b) $\int_{\Phi} \Omega = \int_C \Omega$ Como $\Phi_u\times\Phi_v = N$, Φ es compatible con la orientación de C por N



$\int_C d\omega = \int_{\partial C}\omega = \int_{\phi_1}\omega + \int_{\phi_3}\omega$



$\phi_1(t) = (\cos t, \operatorname{sen} t, 0)$, $0\le t\le 2\pi$

$\phi_3(t) = (\cos t, -\operatorname{sen} t, 1)$, $0\le t\le 2\pi$

$\int_{\phi_1}\omega = \int_0^{2\pi} -(\operatorname{sen} t)^2\,dt = -\int_0^{2\pi}\frac{1-\cos 2t}{2}\,dt = -\pi$

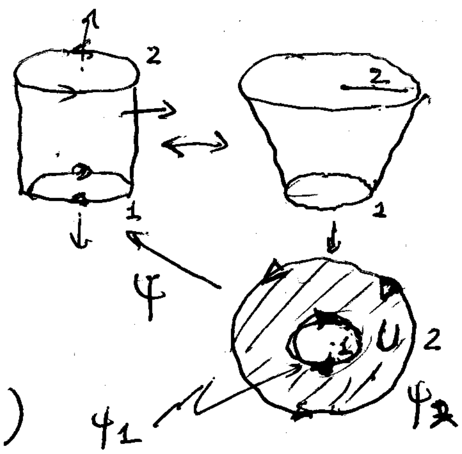
$\int_{\phi_3}\omega = \int_0^{2\pi} e\operatorname{sen}^2 t\,dt - (\cos t)\,dt = e\pi - 0 = e\pi$

$\int_C d\omega = e\pi - \pi = \pi(e-1)$

c) $\varphi : U \to \mathbb{R}^3$, $U = \{1 < u^2+v^2 < 4\}$

$$\varphi(u,v) = \left(\frac{u}{\sqrt{u^2+v^2}}, \frac{v}{\sqrt{u^2+v^2}}, \sqrt{u^2+v^2} - 1 \right)$$

$$\int_C \Omega = -\int_\varphi \Omega$$



$$\varphi_u = \left(\frac{v^2}{(u^2+v^2)^{3/2}}, \frac{-uv}{(u^2+v^2)^{3/2}}, \frac{u}{\sqrt{u^2+v^2}} \right)$$

$$\varphi_v = \left(\frac{-uv}{(u^2+v^2)^{3/2}}, \frac{u^2}{(u^2+v^2)^{3/2}}, \frac{v}{\sqrt{u^2+v^2}} \right)$$

$$\varphi_u \times \varphi_v = \begin{pmatrix} -u/u^2+v^2 \\ -v/u^2+v^2 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{u^2+v^2}} \cdot \begin{pmatrix} -x \\ -y \\ 0 \end{pmatrix} = \frac{1}{\sqrt{u^2+v^2}}(-\vec{N})$$

Entonces $\displaystyle\int_{(C,N)} \Omega = -\int_\varphi \Omega$

Calculo $\displaystyle\int_\varphi d\omega \overset{\text{STOKES}}{=} \int_{\varphi|\partial C} \omega = \int_{①} \varphi^*|_{\partial C}\,\omega + \int_{②} \varphi^*|_{\partial C}\,\omega$

(1+) $\varphi_{1+}(u) = (u, \sqrt{1-u^2}, 0)$ $\quad u^2+v^2=1,$
$-1 \le u \le 1$ $\quad v = \pm\sqrt{1-u^2}$

(1−) $\varphi_{1-}(u) = (-u, -\sqrt{1-u^2}, 0)$
$-1 \le u < 1$

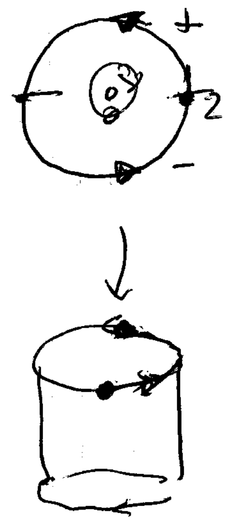
$$\int_{(1+)} \varphi^*|_{\partial C}\,\omega = \int_{-1}^{1} \sqrt{1-u^2}\,du = \int_{-\pi/2}^{\pi/2} \cos^2 t\,dt \quad \begin{pmatrix} u = \operatorname{sen} t \\ du = \cos t\,dt \end{pmatrix}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{1+\cos 2t}{2}\,dt = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

$$\int_{(1-)} \varphi^*|_{\partial C}\,\omega = \int_{-1}^{1} \sqrt{1-u^2}\,du = \frac{\pi}{2}$$

(2+) $\varphi_{(2+)}(u) = \left(-\frac{u}{2}, \frac{\sqrt{4-u^2}}{2}, 1\right), \quad -2 \le u \le 2$

(2−) $\varphi_{(2-)} = \left(\frac{u}{2}, -\frac{\sqrt{4-u^2}}{2}, 1\right), \quad -2 \le u \le 2$



$$\int_{(2+)} \varphi_{(2+)}^{*}\omega = \int_{-2}^{4} e\,\frac{1}{2}\sqrt{4-u^2}\left(-\frac{1}{2}\right)du + \left(-\frac{1}{2}\right)\frac{u}{\sqrt{4-u^2}}\,du$$

$$= -\frac{\pi}{2}e$$

$$\int_{(2-)} \varphi\big|_{(2-)}^{*}\omega = -\frac{\pi}{2}e$$

$$\int_{(C,N)} d\omega = -(\pi - \pi e) = \pi e - \pi = \pi(e-1)$$

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10. $S = \{x^2+y^2+z^2 = 1\}$, $\vec{N}$, $R = [0,2\pi]\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\phi(u,v) = (\cos v\cos u, \cos v \operatorname{sen} u, \operatorname{sen} v)$

$\phi : R \to S$

$\phi_u = (-\operatorname{sen} u\cos v, \cos u\cos v, 0)$

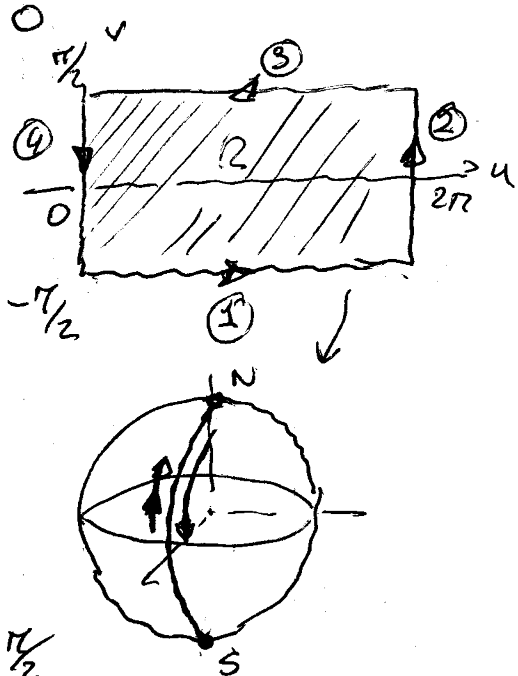
$\phi_v = (-\operatorname{sen} v\cos u, -\operatorname{sen} v\operatorname{sen} u, \cos v)$

$\phi_u \times \phi_v = (\cos^2 v\cos u, \cos^2 v\operatorname{sen} u, \cos v\operatorname{sen} v) = \cos v\,(x,y,z)$

$\cos v > 0$ en $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, ϕ es compatible con (S,N)

b) $\Phi|_{\partial R}$ y $\int_{\partial\Phi} \omega = 0$, ω - 1-forma

Si Ω es 2-forma exacta, $\int_S \Omega = 0$



$$\Phi_1(u) = \Phi(u, -\tfrac{\pi}{2}) = (0,0,-1)$$

$$\Phi_2(v) = \Phi(2\pi, v) = (\cos v, 0, \operatorname{sen} v)$$
$$-\tfrac{\pi}{2} \leq v \leq \tfrac{\pi}{2}$$

$$\Phi_3(u) = \Phi(u, \tfrac{\pi}{2}) = (0,0,1)$$

$$\Phi_4(v) = \Phi(0, v) = (\cos v, 0, \operatorname{sen} v)$$

v comienza en $\pi/2$ y termina en $-\pi/2$

$$\int_{\partial\Phi} \omega = \int_{\partial R} \Phi^*|_{\partial R}\,\omega = 0 + \int_{-\pi/2}^{\pi/2} \Phi_2^*\omega + 0 + \int_{\pi/2}^{-\pi/2} \Phi_4^*\omega = 0$$

$$\int_S \Omega = \int_S d\omega \overset{\text{Stokes}}{=\!=} \int_{\partial\Phi} \omega = 0 \quad \text{pq. } \Omega = d\omega$$

c) $\vec{F} = \rho^{-3}\vec{r} = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}(x,y,z)$ $\qquad \rho = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

$U_0 = \mathbb{R}^3 \setminus \{(0,0,0)\}$ $\quad$ $\operatorname{div}\vec{F} = 0$; $\vec{F}^{\#}$ es cerrada en U^0

$$\frac{\partial}{\partial x}\left(\frac{x}{\|\vec{r}\|^3}\right) = \frac{\|\vec{r}\|^3 - x\,\frac{3}{2}\,\|\vec{r}\| \cdot 2x}{\|\vec{r}\|^6} = \frac{\|\vec{r}\|^3 - 3x^2\|\vec{r}\|}{\|\vec{r}\|^6}$$

$$\left(\sqrt{x^2+y^2+z^2}\right)^3 = (x^2+y^2+z^2)^{3/2}$$

$$\frac{\partial}{\partial y}\left(\frac{y}{\|\vec{r}\|^3}\right) = \frac{\|\vec{r}\|^3 - 3y^2\|\vec{r}\|}{\|\vec{r}\|^6}, \quad \frac{\partial}{\partial z}\left(\frac{z}{\|\vec{r}\|^3}\right) = \frac{\|\vec{r}\|^3 - 3z^2\|\vec{r}\|}{\|\vec{r}\|^6}$$

$$\operatorname{div} \vec{F} = \frac{1}{\|\vec{r}\|^6}\left(3\|\vec{r}\|^3 - 3\|\vec{r}\|(x^2+y^2+z^2)\right) = 0$$

$\vec{F}^{\#}$ es cerrado pq. $d(\vec{F}^{\#}) = (\operatorname{div}\vec{F})\, dx\wedge dy \wedge dz = 0$

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d) $\vec{F}^{\#}$ no es exacta en $U \supset S$

Si $\vec{F}^{\#}$ fuera exacta, $\int_{(S,N)} \vec{F}^{\#} = 0$ (1)

$$\int_{(S,N)} \vec{F}^{\#} = \int_{\Phi} \vec{F}^{\#} = \int_{R} \Phi^{*}(\vec{F}^{\#}) \qquad (2)$$

$\phi(u,v) = (\cos v \cos u, \cos v \operatorname{sen} u, \operatorname{sen} v)$

$$\vec{F}^{\#} = \frac{x}{\|\vec{r}\|^3} dy\wedge dz + \frac{y}{\|\vec{r}\|^3} dz\wedge dx + \frac{z}{\|\vec{r}\|^3} dx\wedge dy$$

$$(2) = \int_{-\pi/2}^{\pi/2}\int_0^{2\pi} (\cos v)\, du\, dv = 2\pi \int_{-\pi/2}^{\pi/2} (\cos v)\, dv$$

$$= 2\pi \left[\operatorname{sen} v\right]_{-\pi/2}^{\pi/2} = 4\pi \neq 0$$

Si existiera $\vec{G}$ tal que $\vec{\operatorname{rot}}(\vec{G}) = \vec{F}$, por el problema 9.9,

$\vec{F}^{\#} = \vec{\operatorname{rot}}(\vec{G})^{\#} = d(\vec{G}^{b})$ e.d. $\vec{F}^{\#}$, sería exacta ...

Pero sabemos que $\vec{F}^{\#}$ no es exacta.

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