Prácticas AM 11/12/2020

1. $\omega = dx\wedge dy + x\,dy\wedge dz$

$\Phi : [0,1]^2 \to \mathbb{R}^3, \quad \Phi(u,v) = (u^2, uv, u+e^v) = (\Phi_1, \Phi_2, \Phi_3)$

$$\Phi^*\omega = d\Phi_1\wedge d\Phi_2 + \Phi_1\, d\Phi_2\wedge d\Phi_3 =$$
$$= (2u)\,du\wedge u\,dv + u^2(v\,du + u\,dv)\wedge(du + e^v dv)$$
$$= 2u^2\,du\wedge dv + u^2 v e^v\,du\wedge dv + u^2 u\,dv\wedge du$$
$$= (2u^2 + u^2 v e^v - u^3)\,du\wedge dv$$

$$\int_\Phi \omega = \int_0^1\int_0^1 \Phi^*\omega = \int_0^1\int_0^1 (2u^2 + u^2 v e^v - u^3)\,du\,dv = \frac{3}{4}$$

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2. a) $\vec{F} = (cx + z e^{yz}, y, -2z) = (F_1, F_2, F_3)$

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = c + 1 - 2 = c - 1 = 0 \Rightarrow c = 1$$

(b) $\vec{F}^{\#} = F_1\,dy\wedge dz + F_2\,dz\wedge dx + F_3\,dx\wedge dy$

$= (x + z e^{yz})\,dy\wedge dz + y\,dz\wedge dx - 2z\,dx\wedge dy$

$d\omega = \vec{F}^{\#}$

$-y\,dx\wedge dz - 2z\,dx\wedge dy$

$a_3(x,y,z) = \int -y\,dx = -yx \;;\; a_2(x,y,z) = \int -2z\,dx = -2zx$

$\eta_1 = -yx\,dz - 2zx\,dy$

$\omega_1 = \vec{F}^{\#} - d\eta_1 = z e^{yz}\,dy\wedge dz$ (no depende de x ni de dx)

$b_3(y,z) = \int z e^{yz}\,dy = e^{yz}; \quad \eta_2 = e^{yz}\,dz$

$\omega_2 = \omega_1 - d\eta_2 = z e^{yz}\,dy\wedge dz - z e^{yz}\,dy\wedge dz = 0$

$\omega_1 = d\eta_2 \Rightarrow \vec{F}^{\#} = d\eta_2 + d\eta_1; \quad \omega = e^{yz}\,dz - yx\,dz - 2zx\,dy$

$\exists\, c \neq 1$ t.q. $d\omega = \vec{F}_c^{\#}$, $\vec{F}_c^{\#}$ sería cerrada. Sabemos que $d\vec{F}_c^{\#} = (\operatorname{div} \vec{F}_c)\, dx \wedge dy \wedge dz = 0$ solo cuando $c = 1$

—— x ——

(c) $\vec{F} = \operatorname{rot}(\vec{G})$; $d\omega = \vec{F}^{\#} = (\operatorname{rot}(\vec{G}))^{\#} \overset{\text{Ej 9.9}}{=} d(\vec{G}^{b})$

$$\omega = \vec{G}^{b} = (-yx + e^{yz})\,dz - 2zx\,dy$$

$$\vec{G} = (0, -2zx, -yx + e^{yz})$$

Si existe $\vec{G}_c$ para otros valores de c, $\vec{F}_c = \operatorname{rot}(\vec{G}_c)$

$$d(\vec{G}_c^{b}) = (\operatorname{rot}(\vec{G}_c))^{\#} = \vec{F}_c^{\#} \Rightarrow d(\vec{F}_c^{\#}) = d(d\vec{G}_c^{b}) = 0$$

$\Rightarrow \vec{F}_c$ cerrada en $\mathbb{R}^3$. Como $c \neq 1$, $\vec{F}_c$ no es cerrada.

—— x ——

(d) $\vec{G} = (2zy, 0, xy + e^{yz})$

$$\vec{G}^{b} = \omega = 2zy\,dx + (xy + e^{yz})\,dz$$

—— x ——

3. $U = \mathbb{R}^2 \setminus \{(0,0)\}$, $\omega = \dfrac{x^2 + cy^2}{(x^2+y^2)^2}(-y\,dx + x\,dy)$

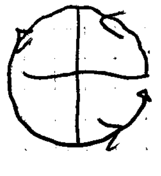
a) ω cerrada

$d\omega = 0 \quad \forall c \in \mathbb{R}$

b) Si $1 + c \neq 0$, ω no es exacta

Si fuera exacta, $\omega = df$, f función $\Rightarrow$ Si ϕ es un camino cerrado $\int_\phi \omega = \int_\phi df = f(\phi(b)) - f(\phi(a)) = 0$

$\phi(t) = (\cos t, \operatorname{sen} t)$, $0 \leq t \leq 2\pi$

 1 $\quad \int_\phi \omega \overset{*}{=} \int_0^{2\pi} \phi^{*}\omega$

$$\omega = \frac{x^2+cy^2}{(x^2+y^2)^2}(-y\,dx + x\,dy) \quad ; \quad \phi(t) = (\cos t, \operatorname{sen} t)$$
$$0 \le t \le 2\pi$$

$$\phi^*\omega = \underbrace{\frac{\cos^2 t + c\operatorname{sen}^2 t}{1}}\left((\operatorname{sen} t)^2 dt + (\cos^2 t)\,dt\right)$$
$$= (\cos^2 t + c\operatorname{sen}^2 t)\,dt$$

$$0 = \int_\phi \omega = \int_0^{2\pi} \phi^*\omega = \int_0^{2\pi} (\cos^2 t + c\operatorname{sen}^2 t)\,dt =$$

$1+c \neq 0$

$$= \int_0^{2\pi}\left(\frac{1+\cos 2t}{2} + c\,\frac{1-\cos 2t}{2}\right)dt = \pi + c\pi = (1+c)\pi \neq 0$$

———— x ————

10.4. $\beta_1 = \{\vec{v}_1, \vec{v}_2\}$, $\beta_2 = \{\vec{v}_1+\vec{v}_2, \vec{v}_1-\vec{v}_2\}$

$M = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$; $\det M = -2$ (Cambia la orientación)

———— x ————

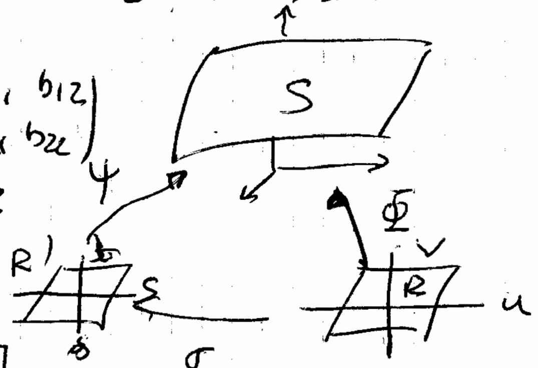
10.5 $\Phi(u,v) : R \to S$, $\sigma(s,t) : R' \to R$

$\psi \equiv \Phi \circ \sigma : R' \to S$, $\psi(s,t) = \Phi(u,v)\big|_{(u,v)=\sigma(s,t)}$

Prueba $\det\left[N \mid \psi_s \mid \psi_t\right] = \det(D\sigma)\cdot\det\left[N \mid \Phi_u \mid \Phi_v\right]$

$$\vec{N} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} \quad A = \begin{array}{c} \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array} \\ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \end{array}, \quad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

3×2 $\qquad$ 2×2



$\det\left[\vec{N}, A\cdot B\right]$

$$= \det\begin{bmatrix} n_1 & a_{11}b_{11} + a_{12}b_{21} \\ n_2 & a_{21}b_{11} + a_{22}b_{21} \\ n_3 & a_{31}b_{11} + a_{32}b_{21} \end{bmatrix} = \det\left[\vec{N} \mid b_{11}\vec{a}_1 + b_{21}\vec{a}_2 \mid b_{12}\vec{a}_1 + b_{22}\vec{a}_2\right]$$

$$\det[\vec{N} \mid A\cdot B] = \det[\vec{N} \mid b_{11}\vec{a}_1 + b_{21}\vec{a}_2 \mid b_{12}\vec{a}_1 + b_{22}\vec{a}_2]$$

$$= \det[\vec{N} \mid \cancel{b_{11}\vec{a}_1 \mid b_{12}\vec{a}_1}]^{0} + \det[N \mid b_{11}\vec{a}_1 \mid b_{22}\vec{a}_2]$$

$$+ \det[\vec{N} \mid b_{21}\vec{a}_2 \mid b_{12}\vec{a}_1] + \det[N \mid \cancel{b_{21}\vec{a}_2 \mid b_{22}\vec{a}_2}]^{0}$$

$$= b_{11}b_{22}\det[N \mid \vec{a}_1 \mid \vec{a}_2] + b_{21}b_{12}\det[\vec{N} \mid \vec{a}_2 \mid \vec{a}_1]$$

$$= (b_{11}b_{22} - b_{21}b_{12})\det[N \mid \vec{a}_1 \mid \vec{a}_2]$$

$$= (\det B)\det[N \mid \vec{a}_1 \mid \vec{a}_2] \qquad (\vec{N} \times (\vec{a}_1 \cdot \vec{a}_2))$$

$$\psi = \Phi \circ \sigma$$

$$\underset{3\times 2}{[\psi_s \mid \psi_t]} = D\psi(s,t) = D\Phi(u,v)\cdot D\sigma(s,t) =$$

$$\underset{3\times 2}{[\Phi_u \mid \Phi_v]} \cdot \underset{2\times 2}{D\sigma(s,t)}$$

$$\det[N \mid \psi_s \mid \psi_t] = \det[N \mid \underbrace{D\Phi(u,v)}_{A} \cdot \underbrace{D\sigma(s,t)}_{B}] =$$

$$(\det D\sigma(s,t))[N \mid \Phi_u \mid \Phi_v] \qquad (5.1)$$

— x —

Supongamos que Φ y ψ son compatibles. Escribir

$$\Omega = F_1\, dy\wedge dz + F_2\, dz\wedge dx + F_3\, dx\wedge dy$$

Se tiene

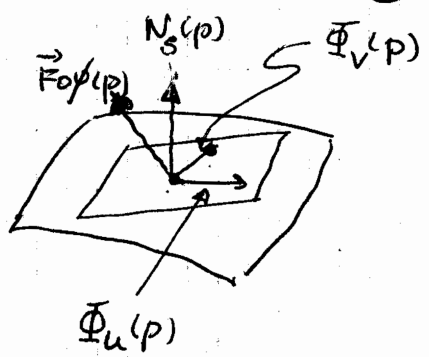
$$\phi^*\Omega = \det\begin{pmatrix} F_1\circ\phi & \frac{\partial\phi_1}{\partial u} & \frac{\partial\phi_1}{\partial v} \\ F_2\circ\phi & \frac{\partial\phi_2}{\partial u} & \frac{\partial\phi_2}{\partial v} \\ F_3\circ\phi & \frac{\partial\phi_3}{\partial u} & \frac{\partial\phi_3}{\partial v}\end{pmatrix} du\wedge dv = \det[\vec{F}\circ\phi \mid \Phi_u \mid \Phi_v]\, du\wedge dv.$$

De manera similar

$$\psi^*\Omega = \det[\vec{F}\circ\phi \mid \psi_s \mid \psi_t]\, ds\wedge dt \qquad (5.2)$$

Entonces

$$\int_\phi \Omega = \int_R \phi^* \Omega = \int_R \det [\vec{F}\circ\phi \mid \Phi_u \mid \Phi_v]\, du\, dv$$



El vector $\vec{F}\circ\phi(p)$ puede descomponerse en un vector normal $\vec{N}_S(p)$ y otro en $T_S(p)$ de la forma $a(p)\,\Phi_u(p) + b(p)\,\Phi_v(p)$. Por las propiedades de los determinantes

$$\det [\vec{F}\circ\phi \mid \Phi_u \mid \Phi_v] = \det [\vec{N}_S \mid \Phi_u \mid \Phi_v]$$

Hacemos el cambio de variables $\sigma : R' \to R$, $\sigma(s,t) = (u,v)$, de manera que

$$du\,dv = \det\left(D\sigma(s,t)\right) ds\,dt$$

Como Φ y ψ son compatibles $\det(D\sigma(s,t)) > 0$ y como ambas parametrizaciones tienen la misma normal $\vec{N}_S$

$$\int_\phi \Omega = \int_R \det [\vec{N}_S \mid \Phi_u \mid \Phi_v]\, du\, dv =$$

$$= \int_{R'} \det [\vec{N}_S \mid \Phi_u\circ\sigma \mid \Phi_v\circ\sigma]\, \det(D\sigma(s,t))\, ds\, dt \overset{(5.1)}{=}$$

Usar que $\vec{N}_S(p) \perp T_p(S)$

$$= \int_{R'} \det [\vec{N}_S \mid \psi_s \mid \psi_t]\, ds\, dt =$$

$$= \int_{R'} \det [\vec{F}\circ\psi \mid \psi_s \mid \psi_t]\, ds\, dt \overset{(5.2)}{=}$$

$$= \int_{R'} \psi^* \Omega = \int_\psi \Omega$$