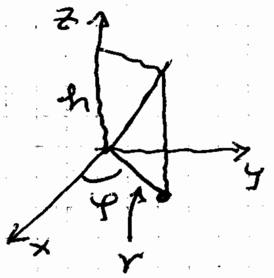
# AM - Prácticas 23-10-2020

T.F. Inversa :

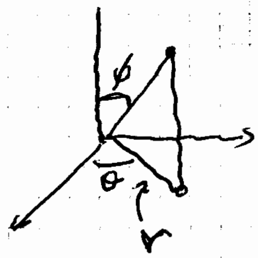
**4.1.**




$$C: \begin{cases} x(r,\varphi,h) = r\cos\varphi \\ y(r,\varphi,h) = r\,\mathrm{sen}\,\varphi \\ z(r,\varphi,h) = h \end{cases}$$

$$DC(r,\varphi,h) = \begin{pmatrix} \cos\varphi & -r\,\mathrm{sen}\,\varphi & 0 \\ \mathrm{sen}\,\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \det(DC(r,\varphi,h))$$

$= r\cos^2\varphi + r\,\mathrm{sen}^2\varphi = r$ . Tiene inversa local cuando $r \neq 0$




$$E: \begin{cases} x(r,\theta,\phi) = r\cos\theta\,\mathrm{sen}\,\phi \\ y(r,\theta,\phi) = r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi \\ z(r,\theta,\phi) = r\cos\phi \end{cases}$$

$$DE(r,\theta,\phi) = \begin{pmatrix} \cos\theta\,\mathrm{sen}\,\phi & -r\,\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi & r\cos\theta\cos\phi \\ \mathrm{sen}\,\theta\,\mathrm{sen}\,\phi & r\cos\theta\,\mathrm{sen}\,\phi & r\,\mathrm{sen}\,\theta\cos\phi \\ \cos\phi & 0 & -r\,\mathrm{sen}\,\phi \end{pmatrix}$$

$\det(DE(r,\theta,\phi)) = -r^2\,\mathrm{sen}\,\phi$ . Tiene inversa cuando $r \neq 0$ y $\mathrm{sen}\,\phi \neq 0$

———— x ————

**5.5.**

$$\begin{aligned} u &= 2x + 2x^2y + 2x^2z + 2xy^2 + 2xyz \\ v &= x + y + 2xy + 2x^2 \\ w &= 4x + y + z + 3y^2 + 3z^2 + 6yz \end{aligned}$$

$F(x,y,z) = (u,v,w)$ $\qquad F: \mathbb{R}^3 \to \mathbb{R}^3$

$$DF(0,0,0) = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix} \Rightarrow \det DF(0,0,0) = 2 \neq 0$$

Existe inversa local cerca de $(0,0,0)$.

(2)

**5.6.** a) $f \in C^1(\mathbb{R})$, $\varepsilon > 0$, $F_\varepsilon(x,y) = (\overbrace{-y+\varepsilon f(x)}^{F_1}, \overbrace{x+\varepsilon f(y)}^{F_2})$.

$(x_0,y_0) \in \mathbb{R}^2$. $F_\varepsilon$ es invertible alrededor de $(x_0,y_0)$

$$DF(x_0,y_0) = \begin{pmatrix} \varepsilon f'(x_0) & -1 \\ 1 & \varepsilon f'(y_0) \end{pmatrix} \Rightarrow \det DF(x_0,y_0) = \varepsilon^2 f'(x_0) f'(y_0) + 1$$

- Si $f'(x_0) = 0$ ó $f'(y_0) = 0 \Rightarrow \det DF(x_0,y_0) = 1 \neq 0$
- Si $f'(x_0) f'(y_0) > 0$, $\det DF(x_0,y_0) > 0$ $(\forall \varepsilon)$
- Si $f'(x_0) f'(y_0) < 0$, $\det DF(x_0,y_0) = \varepsilon^2 f'(x_0) f'(y_0) + 1$

Si $\varepsilon^2 < \dfrac{1}{2|f'(x_0) f'(y_0)|}$, entonces $DF(x_0,y_0) > -\frac{1}{2} + 1 = \frac{1}{2} \neq 0$

—— x ——

(b) $F, G : \mathbb{R}^n \to \mathbb{R}^n$, $c, \lambda$

$\|F(x) - F(y)\| \geq c\|x-y\|$, $\|G(x) - G(y)\| \leq \lambda \|x-y\|$

Sea $H(x) = F(x) + \varepsilon G(x)$. $H$ inyectiva para $\varepsilon$ peq^e.

Sean $H(x) = H(y) \Rightarrow F(x) + \varepsilon G(x) = F(y) + \varepsilon G(y)$

$\Rightarrow F(x) - F(y) = \varepsilon [G(y) - G(x)]$

$c\|x-y\| \leq \|F(x) - F(y)\| = \varepsilon \|G(x) - G(y)\| \leq \varepsilon \lambda \|x-y\|$

$c\|x-y\| \leq \varepsilon\lambda \|x-y\|$. Tomar $\varepsilon \leq \frac{c}{2\lambda}$ o menor

$c\|x-y\| < \frac{c}{2}\|x-y\| \iff 0 \leq \frac{c}{2}\|x-y\| < 0 \Rightarrow \|x-y\| = 0$

$\Rightarrow x = y$

—— x ——

(c) $F_\varepsilon(x,y) = (-y+\varepsilon f(x),\ x+\varepsilon f(y))$

$F(x,y)=(-y,x)$ , $G(x,y)=(f(x),f(y))$

Entonces $F_\varepsilon(x,y)=F(x,y)+\varepsilon G(x,y)$

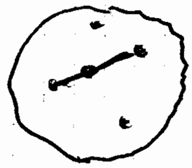
$\|F(x_1,y_1)-F(x_2,y_2)\| = \|(-y_1,x_1)-(-y_2,x_2)\| =$

$= \sqrt{(y_1-y_2)^2+(x_1-x_2)^2} \underset{c=1}{=} \|(x_1,y_1)-(x_2,y_2)\|$

$(x_1,y_1),(x_2,y_2)\in B_\varepsilon(x_0,y_0)$ , $\varepsilon<1$

$\|G(x_1,y_1)-G(x_2,y_2)\| = \|(f(x_1),f(y_1))-(f(x_2),f(y_2))\|$

$=\sqrt{|f(x_1)-f(x_2)|^2+|f(y_1)-f(y_2)|^2} \overset{TVM}{=}$



$=\sqrt{|f'(c_1)|^2|x_1-x_2|^2+|f'(c_2)|^2|y_1-y_2|^2}$

$\le \lambda\sqrt{|x_1-x_2|^2+|y_1-y_2|^2}$ con $\lambda=\max\limits_{(x,y)\in B_1(x_0,y_0)}\{|f'(x)|,|f'(y)|\}$

Elegir $\varepsilon<\frac{1}{2\lambda}$ y también $\varepsilon<1$

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**5.7** $\exists!\ f:U\to\mathbb{R}$, $f\in C^1$ ; $(0,0)\in U$ , $f(0,0)=0$

$e^{f(x,y)}=(1+x\,e^{f(x,y)})(1+y\,e^{f(x,y)})$ , $x,y\in U$

Sea $F(x,y,z)=(1+xe^z)(1+ye^z)-e^z$ , $F(0,0,0)=0$

$\frac{\partial F}{\partial z}(0,0,0)=(xe^z+ye^z+2xye^{2z}-e^z)_{(0,0,0)}=-1\neq0$

El T.F. Implícita da el resultado

**5.8** "Despejar" $u(x,y,z)$ y $v(x,y,z)$ en

$$\left.\begin{array}{l} xy^2+xzu+yv^2=3 \\ xyu^3+2xv-u^2v^2=2 \end{array}\right\}$$

en un entorno de $(1,1,1)=(x,y,z)$ y $(u,v)=(1,1)$

Calcular $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial z}$ en $(1,1,1)$.

Para aplicar TFI, necesito $F: \underset{(x,y,z)}{\mathbb{R}^3} \times \underset{(u,v)}{\mathbb{R}^2} \to \mathbb{R}^2$

$$F(x,y,z,u,v) = (\underbrace{xy^2+xzu+yv^2-3}_{F_1}, \underbrace{xyu^3+2xv-u^2v^2-2}_{F_2})$$

$F(1,1,1,1,1)=(0,0)$

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix}_{(1,1,1,1,1)} = \begin{pmatrix} xz & 2yv \\ 3xyu^2-2uv^2 & 2x-2vu^2 \end{pmatrix}_{(1,1,1,1,1)}$$

$$= \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \Rightarrow \det \frac{\partial F}{\partial(u,v)} = -2 \neq 0$$

$F(x,y,z,u(x,y,z),v(x,y,z))=(0,0)$ en $U \ni (1,1,1)$

$F_1(x,y,z,u(x,y,z),v(x,y,z))=0$ En $(1,1,1,1,1)=p$

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \to$$

$F_2(x,y,z,u(x,y,z),v(x,y,z))=0$

$$\frac{\partial F_2}{\partial x} + \frac{\partial F_2}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial F_2}{\partial v}\cdot\frac{\partial v}{\partial x} = 0 \Rightarrow$$

$$\left\{\begin{array}{l} 2+1\frac{\partial u}{\partial x}(p)+2\frac{\partial v}{\partial x}(p)=0 \\ 3+\frac{\partial u}{\partial x}(p)+0=0 \end{array}\right\}$$

$$\frac{\partial u(p)}{\partial x} = -3 \qquad \frac{\partial v}{\partial x}(p) = \frac{1}{2}$$

Derivando con respecto a $z$ sale

$$\left.\begin{aligned} 1 + \frac{\partial u}{\partial z}(p) + 2\frac{\partial v}{\partial z}(p) &= 0 \\ \frac{\partial u}{\partial z}(p) + 0 \qquad &= 0 \end{aligned}\right\} \quad \frac{\partial u}{\partial z}(p) = 0, \ \frac{\partial v}{\partial z}(p) = -\frac{1}{2}$$

—— x ——

Alba $\displaystyle \underset{2\times 3}{\frac{\partial F}{\partial(x,y,z)}} + \underset{2\times 2}{\frac{\partial F}{\partial(u,v)}} \cdot \underset{2\times 3}{\frac{\partial(u,v)}{\partial(x,y,z)}} = 0_{2\times 3}$

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{pmatrix}_{(1,1,1,1,1)} + \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} & X & \end{pmatrix} = 0$$

$$X = -\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -1 & 0 \\ \frac{1}{2} & -1 & -\frac{1}{2} \end{pmatrix} \qquad \frac{\partial u}{\partial x}(1,1,1) = -3, \quad \frac{\partial v}{\partial z}(1,1,1) = -\frac{1}{2}$$

$$\frac{\partial v}{\partial x}(1,1,1) = \frac{1}{2}$$

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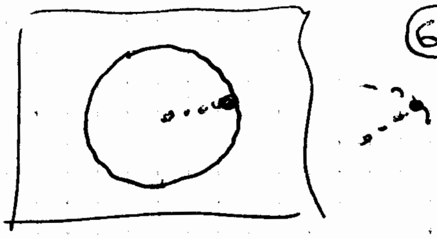
**5.9.** Coerciva $\|f(x) - f(y)\| \geq k\|x - y\| \ \forall x, y$

(a) $f$ coerciva y continua $\Rightarrow$ $f$ cerrada

Sea $C$ cerrado en $\mathbb{R}^n$. Prueba que $f(C)$ es cerrado en $\mathbb{R}^m$

Sea $\{f(x_n)\}_{n=1}^{\infty} \subset f(C)$ convergente en $\mathbb{R}^m$, con



$\lim_{n\to\infty} f(x_n) = y$. Basta probar que $y \in f(C)$.

$$\|x_m - x_n\| \leq \frac{1}{k} \|f(x_m) - f(x_n)\| \xrightarrow[m,n\to\infty]{} 0$$

p.q. $\{f(x_n)\}$ es convergente y por tanto Cauchy.

$\Rightarrow \{x_n\}_{n=1}^{\infty}$ es de Cauchy en $C$. Como $C$ es cerrado

$\lim_{n\to\infty} x_n = x \in C$

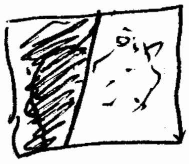
$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = y$$

$\Rightarrow y \in f(C)$

——— x ———

(b) $f : \mathbb{R}^n \to \mathbb{R}^n$ abierta y cerrada $\Rightarrow f$ suprayectiva

Si no fuera suprayectiva, $f(\mathbb{R}^n) \subsetneq \mathbb{R}^n$



$\mathbb{R}^n$ es abierto $\Rightarrow f(\mathbb{R}^n)$ abierto

$\mathbb{R}^n$ es cerrado $\Rightarrow f(\mathbb{R}^n)$ cerrado

$$\mathbb{R}^n = \underbrace{\overset{\neq\emptyset}{f(\mathbb{R}^n)}}_{\text{abierto}} \cup \underbrace{(\mathbb{R}^n - \overset{\neq\emptyset}{f(\mathbb{R}^n)})}_{\text{abierto}}$$

pero $\mathbb{R}^n$ no sería conexo ¡¡¡

——— x ———