Prácticas AM 04-12-2020 ①

1 a) $f: \mathbb{R}^2 \to \mathbb{R}^3$ $\quad f(u_1, u_2) = (u_1^2, u_2^2, e^{u_1 u_2})$

$$\omega = x_2\, dx_1 + (x_1 - x_2 - x_3)\, dx_2 - dx_3$$

$x_1 = u_1^2 \Rightarrow dx_1 = 2u_1\, du_1$

$x_2 = u_2^2 \Rightarrow dx_2 = 2u_2\, u_2$

$x_3 = e^{u_1 u_2} \Rightarrow dx_3 = u_2 e^{u_1 u_2} du_1 + u_1 e^{u_1 u_2} du_2$

$$\omega = (2u_1 u_2^2 - u_2 e^{u_1 u_2})\, du_1 + [2u_2(u_1^2 - u_2^2 - e^{u_1 u_2}) - $$
$$- u_1 e^{u_1 u_2}]\, du_2$$

1 d) $f: \mathbb{R}^2 \to \mathbb{R}^2$, $\quad f(\overset{u}{x}, \overset{v}{y}) = (a\overset{u}{x} - b\overset{v}{y}, b\overset{u}{x} + a\overset{v}{y}) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$\omega = x\, dy - y\, dx$$

$$\begin{aligned}
f^{+}\omega &= (ax - by)\, d(bx + ay) - (bx + ay)\, d(ax - by) \\
&= (ax - by)(b\, dx + a\, dy) - (bx + ay)(a\, dx - b\, dy) \\
&= [(ax - by)b - (bx + ay)a]\, dx \\
&\quad + [(ax - by)a + (bx + ay)b]\, dy \\
&= (-b^2 y - a^2 y)\, dx + (a^2 x + b^2 x)\, dy \\
&= (a^2 + b^2)[x\, dy - y\, dx]
\end{aligned}$$

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3 a) $f: U \to U'$, $\mathcal{C}^2$, ω,

a) ω cerrada $\Rightarrow d\omega = 0$. $\quad d(f^{+}\omega) = f^{*}(d\omega) =$
$= f^{+}(0) = 0$

b) ω exacta $\Rightarrow \exists\, \eta$ t.q. $d\eta = \omega$; $\quad d(f^{+}\eta) = f^{+}(d\eta) = f^{+}(\omega)$
$\Rightarrow f^{+}\omega$ exacta.

4. a) $\omega = (x+y)\,dx + (y-x)\,dy$

$d\omega = d(x+y)\,dx + d(y-x)\,dy = dy\wedge dx - dx\wedge dy$

$= -2\,dx\wedge dy \neq 0$ No es cerrada $\Rightarrow$ No es exacta

b) Sale $d\omega = 0$. Como $\mathbb{R}^2$ es convexo, ω es exacta

Halla $h(x,y,z)$ t.q. $dh = \omega$

$$dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dz$$

$$\frac{\partial h}{\partial x} = y\cos yz \Rightarrow h(x,y,z) = \int y\cos(yz)\,dx = xy\cos yz + g(y,z)$$

$$\frac{\partial h}{\partial y} = \cancel{x\cos yz} - \cancel{xyz\,\mathrm{sen}(yz)} + 2yz = \cancel{x\cos yz} - \cancel{xyz\,\mathrm{sen}(yz)} + \frac{\partial g}{\partial y}$$

$$\therefore \frac{\partial g}{\partial y} = 2yz \Rightarrow g(y,z) = \int 2yz\,dy = zy^2 + f(z)$$

$$h(x,y,z) = xy\cos yz + zy^2 + f(z)$$

$$\frac{\partial h}{\partial z} = (\cancel{y^2} - \cancel{xy^2\,\mathrm{sen}(yz)}) = -\cancel{xy^2\,\mathrm{sen}\,yz} + \cancel{y^2} + f'(z)$$

$$\Rightarrow f'(z) = 0 \Rightarrow f = 0$$

$$h(x,y,z) = xy\cos yz + zy^2$$

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5. Halla $f:\mathbb{R}\to\mathbb{R}$, t.q. $\omega = x^2y\,dx + f(x)\,dy$ exacta en $\mathbb{R}^2$

Sí $0 = d\omega = x^2\,dy\wedge dx + f'(x)\,dx\wedge dy = (f'(x) - x^2)\,dx\wedge dy$

$$\Leftrightarrow f'(x) - x^2 = 0 \Rightarrow f(x) = \frac{x^3}{3}$$

6. $\omega = (1+aze^{yz})\,dx\wedge dy + (1-ye^{yz})\,dx\wedge dz$
$+ (2y+z+\sin z)\,dy\wedge dz$

$0 = d\omega = (ae^{yz} + azye^{yz})\,dz\wedge dx\wedge dy +$
$(-e^{yz} - yze^{yz})\,dy\wedge dx\wedge dz + 0$
$= [a(e^{yz} + zye^{yz}) + (e^{yz} + yze^{yz})]\,dx\wedge dy\wedge dz$

$\Rightarrow a = -1$

Con $a=-1$, ω es la 2-forma del ejercicio 8.11

7. (a) $U\subset\mathbb{R}^n$, $f: U\to\mathbb{R}$; $\phi(t): [a,b]\to U$

$$\int_\phi df = f(\phi(b)) - f(\phi(a))$$

$df = \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}dx_i$, $\int_\phi df = \int_a^b \phi^*(df)\,dt$

$\phi^*(f) = (f\circ\phi)'\,dt \Rightarrow \int_\phi df = \int_a^b (f\circ\phi)'(t)\,dt \overset{TFC}{=}$

$= f\circ\phi(b) - f\circ\phi(a) = f(\phi(b)) - f(\phi(a))$

(b) $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$, $U = \mathbb{R}^2\setminus\{(0,0)\}$ $\phi(t) = (\cos t, \sin t)$
$[a,b] = [0,2\pi]$

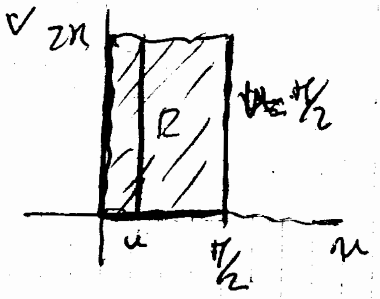
Si fuera exacta, $\int_\phi \omega = \int_\phi df = f(\phi(2\pi)) - f(\phi(0)) = 0$

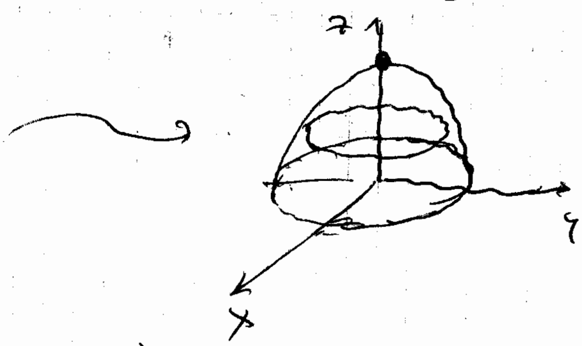
$$\int_\phi \omega = \int_0^{2\pi}\phi^*\omega = \int_0^{2\pi}\underbrace{(\cos t)^2 dt + (\sin t)^2 dt}_{1} = \int_0^{2\pi} dt = 2\pi \neq 0$$

No es exacta en $\mathbb{R}^2\setminus\{(0,0)\}$

b) $\phi(u,v) = (\cos u \cos v, \cos u \operatorname{sen} v, \operatorname{sen} u)$

$R = (0, \pi/2) \times (0, 2\pi)$, $\omega = z\, dx\, dy$





$u = 0 \to (\cos v, \operatorname{sen} v, 0)$

u fija, $\cos^2 u \cos^2 v + \cos^2 u \operatorname{sen}^2 v = \cos^2 u \cdot 1$

$x = \cos u \cos v$, $y = \cos u \operatorname{sen} v$, $0 \le z = \operatorname{sen} u \le 1$

$x^2 + y^2 = \cos^2 u$

$$\int_{\phi(R)} \omega = \int_R \phi^* \omega = \int_0^{\pi/2} \int_0^{2\pi} (\operatorname{sen} u) \left[(-\operatorname{sen} u \cos v\, du - \cos u \operatorname{sen} v\, dv) \right.$$

$$\left. \wedge (-\operatorname{sen} u \operatorname{sen} v\, du + \cos u \cos v\, dv) \right]$$

$$= \int_0^{\pi/2} \int_0^{2\pi} (\operatorname{sen} u) \left[-(\operatorname{sen} u)(\cos u) \cos^2 v - (\operatorname{sen} u)(\cos u) \operatorname{sen}^2 v \right] du\, dv$$

$$= \int_0^{\pi/2} \int_0^{2\pi} -(\operatorname{sen} u)^2 \cos u \; du\, dv = \int_0^{\pi/2} \left[-\frac{(\operatorname{sen} u)^3}{3} \right]_0^{2\pi} dv = 0$$

9. $U \subset \mathbb{R}^3$, $\vec{F} = (F_1, F_2, F_3)$, $\vec{F}^b$, $F^\#$

(a) $(\nabla f)^b = df$

$$\vec{F}^b = F_1\,dx + F_2\,dy + F_3\,dx$$

$$\vec{F}^\# = F_1\,dy \wedge dz + F_2\,dz \wedge dx + F_3\,dx \wedge dy$$

$$\nabla f^b = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)^b = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = df$$

(b) $d(F^\#) = \operatorname{div}(F)\,dx \wedge dy \wedge dz$

$$d(F^\#) = \frac{\partial F_1}{\partial x}dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y}dy \wedge dz \wedge dx + \frac{\partial F_3}{\partial z}dz \wedge dx \wedge dy$$

$$= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz$$

$$= \operatorname{div}(\vec{F})\,dx \wedge dy \wedge dz$$

(c) $d(\vec{F}^b) = (\operatorname{rot} \vec{F})^\#$

$$\operatorname{rot} \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j}$$

$$+ \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

$$d(\vec{F}^b) = \frac{\partial F_1}{\partial y}dy \wedge dx + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy + \frac{\partial F_2}{\partial z}dz \wedge dy$$

$$+ \frac{\partial F_3}{\partial x}dx \wedge dz + \frac{\partial F_3}{\partial y}dy \wedge dz$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)dy \wedge dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dx \wedge dy$$

$$= (\operatorname{rot} F)^\#$$